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(54) Title: CREDIT RISK ESTIMATION SYSTEM AND METHOD

(57) Abstract: A method creates an adjusted credit risk transition matrix and related matrices. The method uses the steps of obtaining a real probability measure (P) transition matrix, and adjusting the (P) transition matrix to be consistent with default probabilities under martingale measure (Q) by making the last column (default probabilities) in the (P) transition matrix consistent with default probabilities calculated under (Q). The method further scales the other entries in the transition matrix (P) to compensate for the adjustment while retaining the relative weights among non-default classes from the (P) transition matrix in the adjusted transition matrix. A credit risk model for the analysis of complex credit instrument implements the method. The adjusted transition matrix may be for use in analysing complex credit instruments. The model may be implemented in computer software on a computer readable medium. The adjusted transition matrix may be used to price credit swaps or value credit instruments. The model may have a simulation engine to generate valuation scenarios based on the adjusted transition matrix. The simulation engine may generate scenarios based on a Monte Carlo method. The model may be extended for the risk analysis of complex portfolios of credit instruments with stochastic interest and foreign exchange rates. The model may have a Monte Carlo simulation engine to generate valuation scenarios based on the joint distribution of interest rates, spreads, foreign exchange rates and obligors' credit migrations consistent with obtained or adjusted (P) transition matrix.

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CREDIT RISK ESTIMATION SYSTEM AND METHOD

FIELD OF THE INVENTION

The field of the invention relates generally to systems and methods for estimating credit risk associated with credit instruments and portfolios of credit instruments.

BACKGROUND OF THE INVENTION

Credit risk models are used to estimate the risk associated with credit instruments and portfolios of credit instruments, such as bonds and loans. Various models have evolved overtime. Many run simulations to generate possible valuation scenarios. Simulations are typically generated using a Monte Carlo method or variant.

Credit risk models typically generate scenarios using probability distributions based on historical data. These probability distributions may be obtained in the form of a transition matrix from various sources, for example Standard & Poors™ and Moody's™. Such probability distributions are often referred to as (*P*), a real measure probability distribution.

In credit risk models, transition matrices under (*P*) probability measure, or (*P*) transition matrices, are used to predict transition, or migration, from one credit class to another, "credit migration", over a given time period. A set of probabilities of migration, or transition, for different credit classes forms a transition matrix.

Models are typically implemented in computer software as part of a credit risk system using compatible hardware.

Alternative matrices, models, systems and methods for estimating credit risk and credit migration associated with credit instruments are desirable.

SUMMARY OF THE INVENTION

In a first aspect the invention provides a method of creating an adjusted (P) transition matrix of probabilities of credit migration for different credit classes. The method uses the steps of

- obtaining a transition matrix under probability measure (P)
- adjusting the (P) transition matrix to be consistent with default probabilities under martingale measure (Q) computed, for instance, from bond market prices by making a column of default probabilities in the (P) transition matrix consistent with default probabilities calculated under (Q)
- scaling the other entries in the (P) transition matrix to compensate for the adjustment while retaining the relative weights among non-default classes from the (P) transition matrix in the adjusted transition matrix.

The method may adjust the (P) transition matrix in accordance with the following:

R = $\{R_1, \ldots, R_K\}$ - are all credit classes

R_K - corresponds to default state

$T^P = p_{i,j}(t_0, t_1)$ - adjusted (P) transition matrix for the time interval $[t_0, t_1]$

$p_{i,j}(t_0, t_1)$ - transition probability from credit class R_i to credit class R_j

$$p^n_{i,j}(t_0, t_1) = \begin{cases} N(z^n_i + C_0), & j = K \\ \upsilon_i \bar{p}^n_{i,j}, & j < K \end{cases} \qquad (i \neq K)$$

$p_{K,j}(t_0, t_1) = 0$ if $j < K$ and $p_{K,K}(t_0, t_1) = 1$

$$C_0 = C_0(G) = \frac{1}{|G|}\sum_{i \in G}(\overline{w}_i - z_i)$$

$$\overline{w}_i = N^{-1}(\overline{p}_{i,K})$$

$$z_i = N^{-1}(q_i)$$

$$\upsilon_i = \frac{1 - N(z_i + C_0)}{1 - \overline{p}_{i,K}}$$

$\overline{p}_{i,K}$	- obtained (P) probability of default for the credit class R_i (from the obtained (P) transition matrix)
q_i	- (Q) probability of default for the credit class R_i
G	- a set of credit classes (can be the whole set $\mathrm{R} = \{R_1, \ldots, R_K\}$ or some subset
$\|G\|$	- denotes the number of elements in G, if G is empty, put $C_0 = 0$
$N(\ )$	- standard normal cumulative density function

Alternatively, the method may adjust the (P) transition matrix in accordance with the following

$T_n^P = \| p_{i,j}^n(t_0, t_1) \|$	- adjusted (P) transition matrix for the time interval $[t_0, t_1]$ for obligor n
$p_{i,j}^n(t_0, t_1)$	- obligor's n transition probability to migrate from credit class R_i to credit class R_j

$$p_{i,j}^n(t_0, t_1) = \begin{cases} N(z_i^n + C_0^n), & j = K \\ \upsilon_i \overline{p}_{i,j}^n, & j < K \end{cases} \qquad (i \neq K)$$

$$p_{K,j}^n(t_0, t_1) = 0 \text{ if } j < K \text{ and } p_{K,K}^n(t_0, t_1) = 1$$

$$C_0^n = C_0^n(t_0, t_1) = \frac{\int_{t_0}^{t_1} (r_s - \alpha_s^n) ds}{\sqrt{var(y_{t_1}^n)}}$$

$$z_i^n = N^{-1}(q_i^n)$$

$$\upsilon_i = \frac{1 - N(z_i^n + C_0^n)}{1 - \bar{p}_{i,K}^n}$$

$\bar{p}_{i,K}^n$ - obtained (P) probability of default for the obligor n over interval $[t_0, t_1]$ if it is in the credit class R_i at time t_0

q_i^n - (Q) probability of default for the obligor n over interval $[t_0, t_1]$ if it is in the credit class R_i at time t_0

r_t - instantaneous risk-free interest rate at time t

$y_{t_1}^n$ - obligor n stock log-return over interval $[t_0, t_1]$

α_t^n - obligor n drift function of time t

In a second aspect the invention provides a credit risk model for the risk analysis of complex portfolios of credit instruments The adjusted (P) transition matrix may be used to value credit swaps and other complex credit instruments The model may have a simulation engine to generate valuation scenarios based on the adjusted (P) transition matrix The simulation engine may generate scenarios based on a Monte Carlo method. The model may be implemented in computer software on a computer readable medium.

In a third aspect the invention provides a method of pricing a credit swap. The method utilizes a number of steps:

- (P) transition matrix over a risk horizon $[t_0, t_1]$ and the (Q) default probabilities over the same period $[t_0, t_1]$ are obtained
- an adjusted (P) transition matrix and (Q) transition matrix over $[t_0, t_1]$ are created
- a first series of forward (Q) transition matrices for standard observation times $t = t_1, t_2, t_3, \ldots$ is built from the (Q) transition matrix and an initial term structure of default probabilities under Q over $[t_0, t_1]$
- the first series is translated to produce a second series, having premium payment dates as the observation times by working with 'fractional' powers of transition matrices for subintervals
- a third series of transition matrices is derived from the second series for co-migration on credit classes for pairs of obligors
- a risk-free discount factor and probabilities contained in the third series are used to discount cash flow for each payment date over the period, and the discounted cash flows are summed for all payment dates. The sum is subtracted from a payoff function for fixed payments to give the credit swap price

In a fourth aspect the invention provides an extended credit risk model for the risk analysis of complex portfolios of credit instruments with stochastic interest and foreign exchange rates. The model may have a Monte Carlo simulation engine to generate valuation scenarios based on the obtained or adjusted (P) transition matrix. The simulation engine may generate scenarios from a joint lognormal distribution of the following risk factors:

- country/industry market indices
- obligors' idiosyncratic components

- base forward rates $f(t_0,t_1), f(t_1,t_2),\dots,f(t_{b-1},t_b)$ for time intervals $[t_0,t_1]$, $[t_1,t_2]$, $\dots$, $[t_{b-1},t_b]$ corresponding to given maturities $t_0,t_1,\dots,t_b$
- credit spread $s_1(t_1)$ for the highest credit class R_1 and for maturity equal to the first bucket point t_1 on the corporate zero curve for each currency
- incremental spreads $\Delta s_i(t_1) = s_i(t_1) - s_{i-1}(t_1)$, $i = 2,\dots,K-1$, for maturity t_1 (by credit class and currency)
- the difference $\Delta s_{K-1}(t_b) = s_{K-1}(t_b) - s_{K-1}(t_1)$ between the spreads for the lowest credit rating for the longest and shortest maturity (by currency)
- foreign exchange rates

From risk factors logreturns for each simulation run the model may compute:

- obligors' credit ratings at risk horizon based on country/industry market index logreturns and idiosyncratic components
- base zero interest rates for all currencies based on forward rates $f(t_0,t_1), f(t_1,t_2),\dots,f(t_{b-1},t_b)$
- credit spreads $s_i(t_1) = s_{i-1}(t_1) + \Delta s_i(t_1)$ for maturity t_1, $i = 2,\dots K-1$ (by currency)
- credit spread $s_{K-1}(t_b) = s_{K-1}(t_1) + \Delta s_{K-1}(t_b)$ for the lowest credit rating and longest maturity t_b (by currency)
- the rest of the spreads for maturity t_b by partitioning the interval $[0, s_{K-1}(t_b)]$ proportionally to $s_i(t_1)$ by taking $s_i(t_b) = s_i(t_1)\dfrac{s_{K-1}(t_b)}{s_{K-1}(t_1)}$ (by currency)
- spreads for other maturities $t_2,t_3,\dots,t_{b-1}$ by linear interpolation between $s_i(t_1)$ and $s_i(t_b)$ (by currency)

- corporate zero curves at risk horizon by adding the appropriate spread to the base curves (by currency)
- foreign exchange rates.

In other aspects the invention provides various models, matrices, software, systems and methods incorporating combinations and subsets of the aspects described above, or utilizing the aspects described above.

BRIEF DESCRIPTION OF THE DRAWINGS

For a better understanding of the present invention and to show more clearly how it may be carried into effect, reference will now be made, by way of example, to the accompanying drawings which show the preferred embodiment of the present invention and in which:

Fig. 1 is a block diagram of the structure of a credit risk model according to the preferred embodiment of the invention,

Fig. 2 is an example of Portfolio Summary (analytic engine was not used) output from the credit risk model of Figure 1,

Fig 3 is an example of Obligor Summary output from the credit risk model of Figure 1,

Fig. 4 is an example of Distribution of the Portfolio Forward Value output from the credit risk model of Figure 1, and

Fig. 5 is an example of Marginal Risk vs. Exposure output from the credit risk model of Figure 1, and

Fig. 6 is a block diagram of the structure of a credit risk model with stochastic interest and foreign exchange rates according to the preferred embodiment of the invention.

DETAILED DESCRIPTION OF THE PREFERRED EMBODIMENTS

Probability distribution (probability measure) (*P*) is a probability distribution of risk factors based on historical data. In credit risk models, probability distribution (*P*) is used to predict transition, or migration, from one credit class to another, "credit migration". A group of probabilities of migration, or transition, for different credit classes forms a transition matrix. A series of transition matrices can be built for different time intervals and for different obligors.

In the preferred embodiment, a given, or obtained, transition matrix under the probability measure (*P*) is adjusted to be consistent with default probabilities under martingale probability measure (*Q*). In the preferred embodiment, the (*P*) transition matrix is adjusted by making a column of default probabilities consistent with default probabilities calculated under (*Q*). The other entries in the transition matrix are scaled to compensate for the adjustment while retaining the relative weights among the non-default classes in the transition matrix.

The adjusted (*P*) transition matrix may be incorporated into a credit risk model to analyze complex portfolios of credit instruments, including credit derivatives. For example, the model may use the adjusted (*P*) transition matrix in place of the given (*P*) transition matrix in

a simulation engine to generate scenarios. The simulation engine may generate scenarios based on any Monte Carlo method.

The model may be implemented in computer software as part of a credit risk system using compatible hardware.

The adjusted (P) transition matrix may be used in the pricing of credit swaps. Credit swap pricing may be incorporated into the credit risk model as it is consistent with the methodology used in the model.

This description has four Parts.

Part 1 describes assignee's basic credit risk model, CreditVaR™, which utilizes deterministic interest and foreign exchange rates. Each obligor's standardized equity returns are decomposed into weighted average of market indices returns (multi-beta model) where the weights are specified to appropriately reflect the obligor's participation in the corresponding markets and to model obligor's idiosyncratic returns. Part 1 has Appendix 1, 2 and 3.

Part 2 describes the derivation of an adjusted (P) transition matrix and related matrices. In particular the relationship between P and Q probabilities for obligors to migrate from one credit class to another including probabilities of default is derived. An adjusted transition matrix and related matrices are created based on the derived relationship. The adjusted (P) transition matrix and related matrices allow portfolios of different instruments, including credit derivatives, to be analyzed. In the preferred embodiment, the adjusted (P) transition matrix and related matrices have been incorporated into an extended version of the credit risk

model of Part 1. Part 2 also describes a method of pricing credit swaps using the adjusted (P) transition matrix and related matrices. In the preferred embodiment, the method has been incorporated into the credit risk model of Part 1. The adjusted transition (P) matrix, related matrices and correlations between equity returns are used to compute joint probability distribution of obligors' credit migrations.

Part 3 describes an extended version of the credit risk model that allows for stochastic interest and foreign exchange rates. The model generates correlated interest rates, foreign exchange rates and obligors' credit migrations according to their joint probability distribution. As stochastic rates are permitted valuation must be performed for each instrument in a generated scenario.

It will be recognized that the adjusted (P) transition matrix and related matrices, pricing method and stochastic features may also be incorporated into other compatible credit risk models.

Part 1 - Basic CreditVaR Methodology

1. Introduction

CreditVaR™ is assignee's proprietary model for measuring and analyzing credit risk in a portfolio context This description is made in reference to CreditVaR as the preferred embodiment of the credit risk model Those skilled in the art will recognise that many of the principles described herein are equally applicable to other credit risk models, such as J.P Morgan's implementation of CreditMetrics™

The methodology used in CreditVaR is based on credit migration analysis and on the assumption that an obligor's credit migration is driven by its asset value For additional discussion of credit migration analysis, please see CreditMetrics™ – Technical Document, April 2, 1997, J P Morgan & Co Incorporated that is hereby incorporated by reference and available at www creditmetrics com

The methodology structure and implementation of CreditVaR are shown in Figure 1 Information required for credit risk analysis includes.

Obligor data:

Information about obligors is organized into a database containing details of their credit ratings, industries, and countries.

Portfolio data:

Information about financial positions is organized into portfolios of exposures It allows to cover different types of instruments such as fixed income instruments, loans, commitments, letters of credit, etc.

Market data, transition probabilities and correlations:

These include yield curves, spread curves, foreign exchange rates, transition probabilities from one credit rating to another for different credit rating systems (Moody 8 states, Moody 18 states, S&P 8 states), correlations between market indices

The CreditVaR methodology may be implemented both analytically and as a Monte Carlo simulation It calculates two risk measures: standard deviation (in the analytic and simulation engines) and percentile level (in the simulation engine) of the portfolio value distribution for a given time horizon The regulatory capital calculation is based on the percentile level, therefore, only simulation engine is used for the capital calculation Accordingly, in the preferred embodiment the analytic engine was removed Following sections describe the methodology and a possible implementation in more details

2. Risk Statistics and Measurement

The CreditVaR methodology can assess the impact of changes in debt value for a given portfolio due to credit quality movements of issuers (obligors) – including downgrades, upgrades and possible defaults – which can occur within the time horizon, typically a period of one year.

There are two computational engines developed in CreditVaR. Analytic Engine (AE) and Simulation Engine (SE). Both are capable of producing *mean* and *standard deviation* for the portfolio's forward value (risk horizon later). The two sets of numbers should be quite comparable as the same modelling framework is embodied in both engines. Though distribution of value for a portfolio of bonds is typically very much skewed, these statistics contribute to some understanding of credit risk present in the portfolio.

The Simulation Engine generates a distribution of forward portfolio values, and therefore has the advantage of estimating any statistic, including ones deemed useful in risk measurement. In particular, *Value-at-Risk* for any percentile level, together with a confidence band, can be computed. (Another risk statistic that can be produced from SE is *average shortfall*, which is the expected loss given that losses exceed a given level. The *expected excession* of a percentile level, which is the expected loss given that the loss is more extreme than the given percentile level, is another very useful risk measurement that can be obtained using SE engine.) Results become more accurate as number of simulations increases, but will exhibit cost in terms of time and memory as with any Monte Carlo method.

With additional computation, both engines AE and SE produce *marginal standard deviation* by each obligor, i.e., the difference between the standard deviation of the entire portfolio and that of the portfolio ex the obligor. Plotting marginal standard deviation expressed as percentage of mean value for the given asset against the exposure size for each obligor, it can be used to identify that part of the portfolio which has concentration of risk, large exposure size and high percentage of marginal standard deviation. All of these measurements are useful in risk monitoring, as well as for capital allocation purpose.

The current CreditVaR method can be extended to produce measurement of risk which are due to obligor-specific returns. Risk due to market indices, in fact to any subset of market indices, can also be obtained

3. Pre-processing

The methodology requires that the three types of data mentioned in Introduction be given as inputs They are then used to prepare the portfolio in an input format applicable to the CreditVaR Engine(s) We now indicate the required pre-processing

For each instrument in the portfolio, there should be sufficient information to calculate its forward value for any credit migration with respect to a credit rating system chosen for CreditVaR Specifications such as coupon rate, term to maturity, yield curve, spreads, etc are needed Recovery rate and its standard deviation, i e. the seniority of the debt, are also required in case default occurs We assume that obligor's exposures can be aggregated or netted across instruments having same seniority

The credit quality movements can be tabulated in a transition matrix (for the given risk horizon) of probabilities for an obligor in one rating to end up in other ratings, including default, or remain unchanged in its rating An additional ingredient essential in the pre-processing is a module which quantitatively measures the portfolio effect of credit by accounting for correlations of asset returns for the obligors Once equipped with these asset correlations (which are on average typically between 20% and 35%), joint rating changes can be modelled across these obligors

Ultimately, it turns out that in this model the only parameters which affect the risk of a portfolio are the two already mentioned, namely, the transition probabilities for each obligor and the correlations between asset returns In this framework, the formulation of risk estimation is then reduced to considering the "standardized" (mean 0, standard deviation 1) asset returns, where the only parameters to be estimated are these correlations.

It should be noted that interest rates and spreads are assumed to be deterministic in this Part 1 In Parts 2 and 3 interest rates and foreign exchange rates are allowed to be stochastic As stochastic rates are permitted valuation must be performed for each instrument in a generated scenario Part 3 describes a method and means for valuing such scenarios.

As asset values are in general difficult to observe, we use instead the correlation between equity returns as its proxy The obvious advantage of this is that equity data are more readily available than data of credit spreads or of actual joint rating changes In order to keep the size of correlation matrix reasonably small to be useful in computation as well, the methodology relies on some form of component analysis Basically, each obligor's standardised asset returns are decomposed into a weighted average of some 'benchmark' country-industry indices where the weights are specified to appropriately reflect the obligor's participation in the corresponding markets, plus a residual component that is associated to obligor's idiosyncratic returns. These weights are specified for each obligor as part of input stored in Obligor database (An example of such specification would be Company X participates 70% in German Chemicals and 30% in German Electronics, and 15% of the movements in X's equity returns are company-specific.) In this way, the methodology incorporates the mix in obligor's market capitalization as well as obligor-specific movements in returns when computing credit risk of the portfolio Given these weights and a correlation

matrix for the indices, one can easily obtain a matrix of correlations between all of the obligors present in the portfolio Appendix 2 presents in detail this methodology

We remark that in applying CreditVaR to a number of portfolios there are several computational modules which can be made external to the pre-processing of individual portfolios For example, the asset correlations between obligors can be stored in a database, and each time a portfolio is encountered for credit risk computation selecting appropriate data can be considered as part of the pre-processing.

4. Analytic Engine (AE)

Mean for the portfolio's forward value can be calculated in a straightforward way, using transition probabilities and forward bond prices which have already been valued in the pre-process Computing standard deviation requires more work, but it reduces to computing standard deviations of various obligor-pair subportfolios. The latter is straightforward as well since joint probabilities of credit quality co-movements are made available in the pre-process Marginal standard deviations can be computed in a similar fashion

5. Simulation Engine (SE)

First, threshold levels of standardised asset returns representing credit rating changes are determined for each obligor in the portfolio using the transition probabilities, this part depends only on the obligor's credit class and not on the obligor itself Using the asset correlations as the covariance matrix, samples of standardised multivariate normal random

variables are generated These are then bucketed using the threshold levels, thereby generating scenarios of credit rating co-movements for the obligors.

For each of these scenarios, portfolio is revalued quickly using pre-processed prices of exposures Each time a default occurs in scenario, i e , when the sampled asset return value of an obligor is below the default threshold level, a random recovery rate is generated according to a beta-distribution whose defining parameters are governed by seniority associated to the obligor's bonds Finally, we obtain a distribution of portfolio values and from it the relevant risk statistics

6. Implementation

A CreditVaR software program utilizes the methodology described above The core computational engine is implemented in C++™. The interface for Windows™ could be a combination of Microsoft Access™/Microsoft Excel™ Windows version will consist of a Microsoft Access/Excel application and a DLL with the compiled computational engine. The selection of hardware is at the discretion of the user provided it has sufficient processing capability and is compatible with the programming environment chosen Other programming environments could be used as would be evident to a person skilled in the art.

CreditVaR program uses a country-industry covariance matrix to compute asset correlations between the obligors in the portfolio Matrices may be contained on a computer readable medium, for example on a hard disk, floppy disk, optical disk, or random access memory, as appropriate for the particular computing environment and stage of use. Matrices may be

contained in an appropriate signal for transmission, such as across the Internet or another computer network

Computation engine consists of 2 parts – analytic and simulation Analytic solution produces total mean and total and marginal standard deviation of every obligor. Simulation produces total mean and standard deviation of every obligor, as well as Value-at-Risk numbers for the whole portfolio. For initial implementation time horizon is taken to be 1 year. The model can be expanded to handle other credit instruments like loans, loan commitments, etc.

Inputs

The current implementation of the Global Analytics CreditVaR model evaluates the credit risk of a bond portfolio based on the following information:

1. Obligor data
   - Current credit ratings of the issuers (obligors), company-specific risk of the obligors, capitalization data
2. Market Data
   - Current zero-rate curves and current spread curves corresponding to different credit ratings and currencies
   - Forward (1 year from now) zero-rate curves and forward spread curves corresponding to different credit ratings and currencies
   - Transition probabilities matrix for credit rating migrations
   - Covariances between the market indices underlying obligors' capitalization
3. Spot and forward (1 year) exchange rates
4. Market Data exposure data (PortEfolio)

- For bonds. principal amount of the bond, coupon rate or spread, seniority (defined by recovery rate and its standard deviation), maturity of the bond, currency, etc
- For loans and loan commitments total authorized amount, loan amount, letters of credit amount. coupon or spread, fees on letters of credit and unused portions, seniority, etc.

Outputs

The program produces the following outputs:

1 Aggregated by obligor:

- Total present value of the bonds by a given obligor
- Total expected 1 year forward value of the bonds (obtained analytically and by simulation)
- Total standard deviation of 1 year forward value of the bonds (obtained analytically and by simulation)
- Stand-alone VaR at different confidence levels
- Marginal standard deviation and marginal risk of a given obligor
- Delta standard deviation and delta VaR

2. Totals for the portfolio:

- Total present value of the portfolio
- Total expected portfolio value in 1 year from now (obtained analytically and by simulation)
- Total standard deviation of the portfolio value in 1 year from now
- Skewness and kurtosis of the portfolio distribution

- Percentiles of the portfolio value (VaR numbers of the portfolio) and expected excession at 0.1%, 0.5%, 1%, 5% and 10% levels
- 95% confidence intervals for the VaR values

Marginal standard deviation and marginal VaR for a given obligor i are defined as

$$STD_{total} - STD_{total \cdot obligor_i},$$

$$VaR_{total} - VaR_{total \cdot obligor_i},$$

where STD_{total}, VaR_{total} - are portfolio standard deviation and VaR, and $STD_{total \cdot obligor_i}$, $VaR_{total \cdot obligor_i}$ - are standard deviation and VaR of the portfolio without positions corresponding to this obligor

Obligor's Delta standard deviation (Delta VaR) is defined as the sensitivity of the portfolio's standard deviation (VaR number) with respect to the aggregated obligor's i position value $p_{obligor_i}$, multiplied by $p_{obligor_i}$

$$DeltaSTD_{obligor_i} = \frac{\partial STD_{total}}{\partial p_{obligor_i}} p_{obligor_i},$$

$$DeltaVaR_{obligor_i} = \frac{\partial VaR_{total}}{\partial p_{obligor_i}} p_{obligor_i}$$

$DeltaSTD_{obligor_i}$ and $DeltaVaR_{obligor_i}$ satisfy following equations [3]:

$$STD_{total} = \sum_i DeltaSTD_{obligor_i},$$

$$VaR_{total} = \sum_i DeltaVaR_{obligor_i}$$

In other words, $DeltaSTD_{obligor_i}$ ($DeltaVaR_{obligor_i}$) is an obligor i contribution into the portfolio's total standard deviation (VaR)

Appendix 1 to Part 1

Sample Input and Output Data

Obligor data table (example) – contains the following information:

- Short and Full name of the obligor
- Current credit rating of the obligor
- Company specific % - the percent of the company's risk that is not explained by company's sensitivity to a particular country or sector of industry

Country 1 (2, 3) and Industry 1 (2, 3) for each country and appropriate percentages – capitalization of the company

Short Name	Long Name	Rating	Company-Specific %	Country 1	Capitalization in Country 1, %	Industry 1	Capitalization in Industry 1, %
AAAAKOT	General Electric	Aaa	20	US	100	GNRL	100
AAAABPD	Hydro-Quebec	Aa	61	CA	100	GNRL	100
AAAAAUD	Ontario, Province Of	Aa	72	CA	100	GNRL	100
28664	Dharmala Sakti Sejahtera, PT	Ba	51	ID	100	GNRL	50
29715	Citicorp	A	53	US	100	GNRL	20
29841	Anwar Sierad, PT	B	51	ID	100	GNRL	100
27508	BOC Group Plc, The	A	52	GB	70	GNRL	50
29842	Dharmala Intl Utama	Ba	51	ID	100	GNRL	100

Short Name	Long Name	Capital-ization in Industry 1,%	Industry 2	Capital-ization in Industry 2,%	Industry 3	Capital-ization in Industry 3,%	Country 2	Capital-ization in Country 2,%
AAAAKOT	General Electric	100						
AAAABPD	Hydro-Quebec	100						
AAAAAUD	Ontario, Province Of	100						
28664	Dharmala Sakti Sejahtera, PT	50	AUTO	20	FOOD	30		
29715	Citicorp	20	BFIN	80				
29841	Anwar Sierad, PT	100	ELCS	50			US	30
27508	BOC Group Plc, The	50						
29842	Dharmala Inti Utama	100						

Current zero-rate curves for computing present values of the bonds (fragment)

Currency	Maturity	Aaa	Aa	A	Baa	Ba	B	Caa
AUD	1	0.053250	0.054766	0.054966	0.056666	0.061166	0.067266	0.077766
AUD	2	0.054404	0.053734	0.054536	0.056133	0.062178	0.067867	0.081702
AUD	3	0.057863	0.056439	0.057031	0.058845	0.065964	0.072324	0.095020
AUD	5	0.062668	0.059883	0.060592	0.062078	0.071337	0.078535	0.109867
AUD	7	0.065801	0.062786	0.063617	0.065720	0.075229	0.084363	0.125788
AUD	10	0.068452	0.065865	0.066699	0.069597	0.079848	0.090017	0.145838
AUD	15	0.070039	0.068845	0.069800	0.072824	0.084333	0.094863	0.160516
AUD	20	0.070246	0.077058	0.078239	0.081695	0.095994	0.108289	0.160516
AUD	30	0.068616	0.075739	0.077275	0.081163	0.100634	0.115712	0.160516
BEF	1	0.044700	0.040700	0.041400	0.043100	0.047600	0.053700	0.064200
BEF	2	0.047977	0.042845	0.043648	0.045248	0.051292	0.056991	0.070827
BEF	3	0.051217	0.045657	0.046252	0.048066	0.055175	0.061536	0.084151
BEF	5	0.055735	0.050484	0.051195	0.052688	0.061934	0.069137	0.100376
BEF	7	0.059462	0.055583	0.056418	0.058528	0.068088	0.077254	0.118753
BEF	10	0.063016	0.059309	0.060148	0.063046	0.073351	0.083559	0.139378
BEF	15	0.066228	0.062416	0.063372	0.066395	0.077919	0.088479	0.153885
BEF	20	0.068648	0.065104	0.066209	0.069446	0.082701	0.094053	0.187946
BEF	30	0.062548	0.069235	0.070766	0.074696	0.094125	0.109529	0.187946
CAD	1	0.046084	0.051908	0.052608	0.054308	0.058808	0.064908	0.075408
CAD	2	0.048069	0.051797	0.052600	0.054197	0.060242	0.065933	0.079768
CAD	3	0.051818	0.053409	0.054000	0.055812	0.062919	0.069272	0.091916
CAD	5	0.055428	0.055384	0.056091	0.057574	0.066785	0.073954	0.105090
CAD	7	0.058771	0.057811	0.058638	0.060727	0.070182	0.079251	0.120275
CAD	10	0.062437	0.060102	0.060929	0.063788	0.073943	0.083994	0.138775
CAD	15	0.067168	0.064606	0.065561	0.068582	0.080094	0.090640	0.156078
CAD	20	0.069212	0.066522	0.067619	0.070830	0.083975	0.095206	0.187307
CAD	30	0.070499	0.067655	0.069103	0.072814	0.090877	0.104848	0.187307
CHF	1	0.025700	0.018100	0.018800	0.020500	0.025000	0.031100	0.041600
CHF	2	0.029251	0.021334	0.022137	0.023738	0.029770	0.035476	0.049285
CHF	3	0.029647	0.024503	0.025101	0.026913	0.033995	0.040351	0.062777
CHF	5	0.029943	0.030250	0.030960	0.032465	0.041641	0.048827	0.079720
CHF	7	0.030472	0.035345	0.036174	0.038262	0.047756	0.056809	0.097483
CHF	10	0.031012	0.040189	0.041027	0.043866	0.054116	0.064209	0.118510
CHF	15	0.031123	0.044627	0.045579	0.048570	0.060025	0.070555	0.134308
CHF	20	0.031259	0.047376	0.048456	0.051639	0.064593	0.075804	0.162180
CHF	30	0.031582	0.055377	0.056900	0.060904	0.080289	0.096276	0.162180
DEM	1	0.043600	0.040200	0.040900	0.042600	0.047100	0.053200	0.063700
DEM	2	0.045953	0.041731	0.042534	0.044133	0.050175	0.055873	0.069703
DEM	3	0.050143	0.045279	0.045874	0.047689	0.054801	0.061166	0.083789
DEM	5	0.054534	0.049986	0.050697	0.052191	0.061435	0.068639	0.099872
DEM	7	0.058249	0.054607	0.055441	0.057548	0.067092	0.076240	0.117637
DEM	10	0.061660	0.058466	0.059305	0.062197	0.072494	0.082689	0.138390
DEM	15	0.065009	0.062601	0.063563	0.066605	0.078204	0.088840	0.154962
DEM	20	0.066847	0.064869	0.065974	0.069215	0.082483	0.093849	0.187930
DEM	30	0.070431	0.074380	0.076066	0.080398	0.102531	0.120894	0.187930

Forward (1 year) zero-rate curves for computing forward values of the bonds (fragment):

Currency	Maturity	Aaa	Aa	A	Baa	Ba	B	Caa
AUD	0	0.055559	0 053202	0.054107	0 055601	0.063192	0 068470	0 085654
AUD	1	0.055559	0.053202	0.054107	0.055601	0 063192	0 068470	0 085654
AUD	2	0.060177	0.057528	0.058066	0.059937	0.068371	0 074863	0 103750
AUD	4	0.065036	0.061292	0.062003	0.063435	0.073895	0 081370	0 118040
AUD	6	0.067908	0.064212	0.065066	0 067236	0.077591	0 087239	0 133997
AUD	9	0.070155	0.067162	0 068010	0 071044	0 081944	0.092575	0 153663
AUD	14	0.071248	0 069894	0 070867	0.073988	0.086007	0.096861	0.166664
AUD	19	0.071148	0.078271	0.079478	0.083029	0 097859	0 110491	0.165043
AUD	29	0 069150	0 076488	0 078053	0 082017	0.102020	0 117422	0 163480
BEF	0	0.051264	0 044994	0.045901	0.047400	0.054998	0.060293	0 077496
BEF	1	0.051264	0 044994	0.045901	0 047400	0 054998	0.060293	0.077496
BEF	2	0.054490	0 048145	0 048686	0 050557	0.058983	0 065477	0.094266
BEF	4	0 058511	0.052945	0.053657	0 055099	0 065547	0 073032	0.109610
BEF	6	0.061942	0 058084	0.058942	0 061122	0 071542	0 081230	0.128113
BEF	9	0.065071	0.061397	0 062252	0.065285	0 076252	0 086928	0.148053
BEF	14	0.067782	0 063984	0 064959	0.068079	0.080118	0 091007	0 160574
BEF	19	0.069923	0 066404	0 067530	0.070851	0 084581	0.096219	0 194843
BEF	29	0 063169	0 070233	0 071793	0 075803	0 095765	0 111506	0 192460
CAD	0	0.050057	0 051687	0 052592	0.054087	0 061679	0 066959	0 084146
CAD	1	0 050057	0.051687	0.052592	0.054087	0 061679	0.066959	0.084146
CAD	2	0.054696	0.054160	0 054697	0 056566	0.064981	0.071461	0.100264
CAD	4	0 057776	0 056255	0 056963	0 058392	0 068789	0 076227	0 112638
CAD	6	0.060901	0 058799	0 059647	0.061801	0.072090	0 081660	0.127933
CAD	9	0 064270	0.061017	0.061858	0 064847	0 075638	0 086136	0 146042
CAD	14	0.068690	0 065519	0.066492	0 069609	0.081631	0.092501	0 162067
CAD	19	0.070443	0.067297	0.068415	0.071707	0.085316	0 096824	0.193509
CAD	29	0 071351	0 068202	0.069677	0.073458	0.092000	0.106251	0 191367
CHF	0	0.032815	0.024579	0.025484	0 026986	0.034562	0.039871	0 057028
CHF	1	0.032815	0 024579	0.025484	0 026986	0 034562	0.039871	0 057028
CHF	2	0.031626	0.027719	0 028266	0.030135	0.038522	0 045007	0 073527
CHF	4	0.031006	0.033310	0 034023	0 035479	0 045844	0.053306	0 089466
CHF	6	0.031270	0 038247	0 039099	0 041253	0.051598	0 061155	0 107084
CHF	9	0.031604	0 042673	0.043527	0 046495	0.057401	0 067953	0 127399
CHF	14	0.031511	0 046548	0 047518	0 050605	0 062572	0 073430	0.141238
CHF	19	0 031553	0 048939	0 050041	0.053304	0 066718	0.078210	0 168899
CHF	29	0 031785	0 056686	0.058239	0 062326	0.082248	0 098595	0 166578
DEM	0	0 048311	0.043265	0 044171	0 045669	0.053260	0 058552	0 075739
DEM	1	0.048311	0 043265	0 044171	0.045669	0 053260	0.058552	0.075739
DEM	2	0.053430	0 047828	0.048370	0.050243	0 058673	0 065171	0.093976
DEM	4	0.057285	0.052447	0.053160	0 054603	0.065050	0 072535	0 109106
DEM	6	0.060711	0.057027	0 057885	0.060060	0.070461	0 080129	0.126889
DEM	9	0.063686	0 060515	0 061369	0 064398	0.075353	0 086016	0 147006
DEM	14	0.066555	0.064220	0 065201	0 068341	0.080461	0.091432	0 161773
DEM	19	0.068085	0 066183	0.067310	0.070635	0.084379	0.096031	0.194856
DEM	29	0.071368	0075578	0.077299	0.081726	0.104493	0 123304	0 192463

Note forward curves were obtained from the current curves by bootstrapping method

Table of foreign exchange rates (spot and forward):

Currency	USD per Ccy, spot	USD per Ccy, fwd 1 yr
ATS	0.0812843	0.0811934
AUD	0.7000000	0.6672129
BEF	0.0277393	0.0276916
CAD	0.7058159	0.6984693
CHF	0.7041758	0.7163023
DEM	0.5720824	0.5717753
DKK	0.1502765	0.1495353
ESP	0.0067732	0.0066965
FIM	0.1895950	0.1886958
FRF	0.1708701	0.1706838
GBP	0.6898000	1.6363808
HKD	0.1293494	0.1221799
IDR	0.0002805	0.0000571
IEP	1.4936000	1.4107878
ITL	0.0005838	0.0005726
JPY	0.0079494	0.0083655
MXN	0.1222046	0.1055812
MYR	0.2928258	0.2089619
NLG	0.5076400	0.5070257
NOK	0.1407856	0.1371640
NZD	0.6266000	0.5710057
PTE	0.0056032	0.0055445
SEK	0.1317020	0.1275255
SGD	0.6327912	0.5657095
THB	0.0252048	0.0166732
USD	1.0000000	1.0000000
XEU	1.1342000	1.1190554
ZAR	0.2069536	0.1866202

Covariance matrix of country-industry indices (fragment, actual size 211 by 211)

Covariance matrix						
		Country	AU	AU	AU	AU
Index name	Country	industry	AU_GNRL	AU_BFIN	AU_BMED	AU_CSTR
MSCI Australia Index (.CIAU)	AU	AU_GNRL	0.0002008			
ASX Banks & Finance Index (.ABII)	AU	AU_BFIN	0.0001658	0.0003709		
ASX Media Index (.AMEI)	AU	AU_BMED	0.0002384	0.0001444	0.0005618	
ASX Building Materials Index (.ABMI)	AU	AU_CSTR	0.000171	8.437E05	0.000192	0.0004216
ASX Chemicals Index (.ACII)	AU	AU_CHEM	0.0002567	0.0001624	0.0003524	0.00023
ASX Energy Index (.AXEY)	AU	AU_ENRG	0.0001518	0.0001079	0.0001871	0.0001241
ASX Food & Household Goods Index (.AFHI)	AU	AU_FOOD	0.0002046	0.0001923	0.0002408	0.0001694
ASX Insurance Index (.AIII)	AU	AU_INSU	0.0002008	0.000195	0.000256	0.0001246
ASX Paper & Packaging Index (.APPI)	AU	AU_PAPR	0.0001811	7.415E-05	0.0002756	0.0002049
ASX Transport Index (.ATII)	AU	AU_TRAN	0.0001754	0.0001389	0.00025	0.0001775
MSCI Austria Index (.CIAT)	AT	AT_GNRL	5.911E-05	9.668E-06	0.0001002	7.506E-05
MSCI Belgium Index (.CIBE)	BE	BE_GNRL	8.795E-05	5.433E-05	0.000145	9.354E-05
MSCI Canada Index (.CICA)	CA	CA_GNRL	0.0001152	0.000125	0.000127	7.762E-05
Toronto SE Automobiles & Parts Index (.TIPA)	CA	CA_AUTO	7.868E-05	8.753E-05	8.144E-05	4.668E-05
Toronto SE Financial Services Index (.TFS)	CA	CA_BFIN	0.0001036	0.0001805	0.0001739	0.0001015
Toronto SE Broadcasting Index (.TCMB)	CA	CA_BMED	4.429E-05	1.239E-05	4.918E-05	6.247E-05
Toronto SE Cement & Concrete Index (.TIPC)	CA	CA_CSTR	0.0001121	7.239E-05	0.0001713	8.249E-05
Toronto SE Chemicals Index (.TIPZ)	CA	CA_CHEM	0.000119	0.0001422	0.0001386	8.82E-05

Transition probabilities of credit rating migrations (Moody-8 rating system):

Rating	Aaa	Aa	A	Baa	Ba	B	Caa	Default
Aaa	0.9338	0.0594	0.0064	0.0000	0.0002	0.0000	0.0000	0.0002
Aa	0.0161	0.9055	0.0746	0.0026	0.0009	0.0001	0.0000	0.0002
A	0.0007	0.0228	0.9242	0.0463	0.0045	0.0012	0.0001	0.0002
Baa	0.0005	0.0026	0.0551	0.8848	0.0476	0.0071	0.0008	0.0015
Ba	0.0002	0.0005	0.0042	0.0516	0.8691	0.0591	0.0024	0.0129
B	0.0000	0.0004	0.0013	0.0054	0.0635	0.8422	0.0191	0.0681
Caa	0.0000	0.0000	0.0000	0.0062	0.0205	0.0408	0.6919	0.2406
Default	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

Note: Any transition probability matrix may be used (not necessarily 8-state).

Example 1

Sample bond portfolio.

Exposure Name	Obligor Name	Rating	Currency	Recovery Rate
912810EN4-11	AAAAKOT	Aaa	USD	40
021654878440001	AAAAAUD	Aa	CAD	50
19971027-00539	AAAAKOT	Aaa	USD	40
027054867160001	AAAABPD	Aa	CAD	60
027052668030001	AAAABPD	Aa	CAD	38
021655365450002	AAAAAUD	Aa	CAD	32
021655355640001	AAAAAUD	Aa	CAD	17

Exposure Name	RecovRate StdDev	Amount	Maturity	Rate	Freq-uency
912810EN4-11	25	5000000.00	15/11/2022	7.625	2
021654878440001	25	15151000.00	11/03/2003	8	2
19971027-00539	25	14000000.00	15/08/2027	6.375	2
027054867160001	30	4905000.00	15/08/2020	11	2
027052668030001	23	1300000.00	15/10/2004	11	2
021655365450002	20	25000000.00	02/06/2027	7.6	2
021655355640001	11	15420000.00	01/12/2005	8.25	2

"Rate" is the fixed coupon rate of the bond. "Frequency" is coupon payment frequency per year Obligors

AAAAKOT - General Electric

AAAABPD - Hydro-Quebec

AAAAAUD - Ontario, Province Of

Sample output (obtained by running CreditVaR program on the above portfolio)

```
Portfolio: Exp7

Number of simulation runs: 100000

Number of exposures: 7

Number of obligors: 3

Evaluation date: 14/04/98

Started: 11:14:37

Finished: 11:16:24

Simulation time: 00:01:47

Analytic solution time: 00:00:00
```

Portfolio Present Value = 71786803.00

Portfolio Mean (analytic) = 70144493.80

Portfolio StDev (analytic) = 523028.46

Portfolio Mean (simulation) = 70143025.60

Portfolio StDev (simulation) = 530826.76

Portfolio Percentile (10%) = -105932.78

Portfolio Percentile (5%) = -297306.87

Portfolio Percentile (1%) = -812778.79

Portfolio Percentile (0.5%) = -970228.53

Portfolio Percentile (0.1%) = -4912661.85

Obligor	Rating	Present Value	Mean (analytic)	Mean (simulation)
AAAAKOT	Aaa	19190259.30	19093143.19	19093014.25
AAAAAUD	Aa	45880465.72	44557820.87	44556843.65
AAAABPD	Aa	6716077.99	6493529.74	6493167.70

Obligor	St.Dev (analytic)	St.Dev. (simulation)	Marginal St.Dev	Marginal Risk
AAAAKOT	73207.89	73760.89	3636.56	0.019%
AAAAAUD	507005.01	515008.37	422137.89	0.920%
AAAABPD	71014.55	78555.92	12079.87	0.180%

Example 2 (actual program output)

Exposure data (fragment of input, actual size is 1863 exposures, 207 obligors)

ID	Exposure Name	Obligor Short Name	Credit Rating	Currency	Recovery Rate	Recovery StDev	Amount	Maturity	Rate	Frequency
15	15656	313122	Aa	AUD	40	25	1000000	3/20/08	6.4	2
16	15657	313122	Aa	AUD	40	25	1000000	3/20/13	6.575	2
17	15658	313122	Aa	AUD	40	25	1000000	3/20/18	7	2
18	15659	313122	Aa	AUD	40	25	1000000	3/20/28	7.05	2
19	15660	313123	A	AUD	40	25	1000000	3/20/99	5.41	2
20	15661	313123	A	AUD	40	25	1000000	3/20/00	5.41	2
21	15662	313123	A	AUD	40	25	1000000	3/20/01	5.64	2
22	15663	313123	A	AUD	40	25	1000000	3/20/03	6	2
23	15664	313123	A	AUD	40	25	1000000	3/20/05	6.27	2
24	15665	313123	A	AUD	40	25	1000000	3/20/08	6.53	2
25	15666	313123	A	AUD	40	25	1000000	3/20/13	6.71	2
26	15667	313123	A	AUD	40	25	1000000	3/20/18	7.14	2
27	15668	313123	A	AUD	40	25	1000000	3/20/28	7.2	2
28	15669	313124	Baa	AUD	40	25	1000000	3/20/99	5.61	2
29	15670	313124	Baa	AUD	40	25	1000000	3/20/00	5.6	2

Example of the program output - Portfolio Summary (analytic engine was not used)

(see Figure 2)

Example of the program output - Obligor Summary

(see Figure 3)

Example of the program output - Distribution of the portfolio forward value

(See Figure 4)

Example of the program output - Marginal Risk vs. Exposure

(See Figure 5)

Appendix 2 to Part 1

Correlation model

2.1 Market and specific components and country/industry weights

Let us assume that company's standardized equity return r of the firm A can be expressed in the form

$$r = \beta_M r_M + \beta_S r_S, \qquad (1)$$

where r_M is standardized return of the market component of r and r_S is standardized return of the idiosyncratic component. r_M and r_S are independent From this follows that coefficients β_M and β_S satisfy condition

$$\beta_M^2 + \beta_S^2 = 1 \qquad (2)$$

We also assume that return of the market component r_M is linear combination of standardized country/industry index returns r_i, which are independent of r_S:

$$r_M = \beta_1 r_1 + \ldots + \beta_n r_n. \qquad (3)$$

$$cor(r_i, r_S) = 0, \; i = 1,\ldots,n$$

Equations (1), (3) can be obtained by regressing A equity returns against index returns in a usual way.

There is another way to represent r in terms of index returns r_i and idiosyncratic return r_S. This representation is consistent with (1)-(3) but based on country/industry weighting coefficients We show below how this representation can be obtained from (1)-(3)

Denote by σ, σ_i - standard deviations of the firm A equity return and index i return respectively, ρ_{ij} correlation between index returns r_i and r_j $(i, j = 1,\ldots,n)$. Then from (3) one has

$$
\begin{aligned}
r_M &= \beta_1 r_1 + \ldots + \beta_n r_n \\
&= \frac{\beta_1}{\sigma_1}\sigma_1 r_1 + \ldots + \frac{\beta_n}{\sigma_n}\sigma_n r_n \\
&= \left(\frac{\beta_1}{\sigma_1} + \ldots + \frac{\beta_n}{\sigma_n}\right)\left(\frac{\beta_1}{\sigma_1\left(\frac{\beta_1}{\sigma_1} + \ldots + \frac{\beta_n}{\sigma_n}\right)}\sigma_1 r_1 + \ldots + \frac{\beta_n}{\sigma_n\left(\frac{\beta_1}{\sigma_1} + \ldots + \frac{\beta_n}{\sigma_n}\right)}\sigma_n r_n\right)
\end{aligned}
$$

or

$$
r_M = \left(\frac{\beta_1}{\sigma_1} + \ldots + \frac{\beta_n}{\sigma_n}\right)\left(w_1\sigma_1 r_1 + \ldots + w_n\sigma_n r_n\right), \tag{4}
$$

where $w_1, w_2, \ldots, w_n$ are given by

$$
w_i = \frac{\beta_i}{\sigma_i\left(\frac{\beta_1}{\sigma_1} + \ldots + \frac{\beta_n}{\sigma_n}\right)}, \quad i = 1,\ldots,n. \tag{5}
$$

Note, that

$$
w_1 + w_2 + \ldots + w_n = 1.
$$

Coefficients $w_1, w_2, \ldots, w_n$ are called country/industry weights and can be expressed in percents.

From equation (4) follows that

$$
var(r_M) = \left(\frac{\beta_1}{\sigma_1} + \ldots + \frac{\beta_n}{\sigma_n}\right)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_i \sigma_j \rho_{ij} = 1. \tag{6}
$$

Denote

$$\hat{\sigma} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_i \sigma_j \rho_{ij} \right)^{\frac{1}{2}} \tag{7}$$

Then from (6) we have

$$\left(\frac{\beta_1}{\sigma_1} + \ldots + \frac{\beta_n}{\sigma_n} \right) = \frac{1}{\hat{\sigma}} \tag{8}$$

Using (1), (4), (8) one can express standardized equity return in terms of country/industry weights as

$$\begin{aligned} r &= \beta_{Mt} r_{Mt} + \beta_S r_S \\ &= \beta_{Mt} \left(\frac{w_1 \sigma_1}{\hat{\sigma}} r_1 + \ldots + \frac{w_n \sigma_n}{\hat{\sigma}} r_n \right) + \beta_S r_S \end{aligned} \tag{9}$$

Hence, to compute this representation one needs to have following parameters: country/industry weights, index standard deviations and correlations, specific coefficient β_S (coefficient β_{Mt} for the market component can be computed from (2) as $\beta_{Mt} = \sqrt{1 - \beta_S^2}$). In the current implementation CreditVaR I derives parameters β_S and β_{Mt} from "specific risk percent" s defined as a fraction of total equity return changes explained by firm-specific movements:

$$\beta_{Mt} = \sqrt{1 - s^2},$$

$$\beta_S = s \tag{10}$$

2.2 Correlation between two obligors

Equations (1), (3) as well as equation (9) allow to compute correlations between equity returns of any two firms.

Let r^B, r^C are standardized equity returns of two different firms B and C respectively and

$$r^B = \beta_M^B \left(\frac{w_1^B \sigma_1}{\hat{\sigma}} r_1 + \ldots + \frac{w_n^B \sigma_n}{\hat{\sigma}} r_n \right) + \beta_S^B r_S^B = \widetilde{\beta}_1^B r_1 + \ldots + \widetilde{\beta}_n^B r_n + \beta_S^B r_S^B ,$$

$$r^B = \beta_M^C \left(\frac{w_1^C \sigma_1}{\hat{\sigma}} r_1 + \ldots + \frac{w_n^C \sigma_n}{\hat{\sigma}} r_n \right) + \beta_S^C r_S^C = \widetilde{\beta}_1^C r_1 + \ldots + \widetilde{\beta}_n^C r_n + \beta_S^C r_S^C ,$$

where

$$\hat{\sigma}_B = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} w_i^B w_j^B \sigma_i \sigma_j \rho_{ij} \right)^{\frac{1}{2}} , \quad \hat{\sigma}_C = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} w_i^C w_j^C \sigma_i \sigma_j \rho_{ij} \right)^{\frac{1}{2}} \qquad (11)$$

$$\widetilde{\beta}_i^B = \beta_M^B \frac{w_i^B \sigma_i}{\hat{\sigma}_B} , \quad \widetilde{\beta}_i^C = \beta_M^C \frac{w_i^C \sigma_i}{\hat{\sigma}_C} , \; i = 1, \ldots, n \qquad (12)$$

Then correlation between r^B and r^C is

$$cor\left(r^B, r^C\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \widetilde{\beta}_i^B \widetilde{\beta}_j^C \rho_{ij} \qquad (13)$$

Example.

Suppose we wish to compute correlation between two obligors B and C in the obligor table in Appendix 1 (Sample Input and Output Data). Obligors short names are "28664" and "29715" respectively. From the table we have

Obligor B ("28664")

Specific percent $s^B = 0.87$

Country/industry weights $w^B_{ID.GNRL} = 0.5$, $w^B_{ID.AUTO} = 0.2$, $w^B_{ID.FOOD} = 0.3$

Obligor C ("29715")

Specific percent $s^C = 0.88$

Country/industry weights $w^{C}_{US\,GNRL} = 0.2$, $w^{C}_{US,BFIN} = 0.8$.

Let us assume following standard deviations and correlations between market indices.

Index	Standard Deviation	Correlations				
		ID GNRL	ID AUTO	ID FOOD	US GNRL	US BFIN
Indonesia, General	0.03	1	0.4	0.2	0.01	0.03
Indonesia, Automobiles	0.02	0.4	1	0.1	0.02	0.04
Indonesia, Food	0.04	0.2	0.1	1	0.01	0.02
US, General	0.01	0.01	0.02	0.01	1	0.3
US, Banking and Finance	0.02	0.03	0.04	0.02	0.3	1

According to (7)

$$\begin{aligned}\hat{\sigma}_B^2 &= 0.03^2 \times 0.5^2 + 0.02^2 \times 0.2^2 + 0.04^2 \times 0.3^2 + 2 \times 0.03 \times 0.02 \times 0.5 \times 0.2 \times 0.4 \\ &\quad + 2 \times 0.03 \times 0.04 \times 0.5 \times 0.3 \times 0.2 + 2 \times 0.02 \times 0.04 \times 0.2 \times 0.3 \times 0.1 \\ &= 0.000471\end{aligned}$$

$$\hat{\sigma}_B = \sqrt{0.000471} = 0.022$$

$$\begin{aligned}\hat{\sigma}_C^2 &= 0.01^2 \times 0.2^2 + 0.02^2 \times 0.8^2 + 2 \times 0.01 \times 0.02 \times 0.2 \times 0.8 \times 0.3 \\ &= 0.000279\end{aligned}$$

$$\hat{\sigma}_C = \sqrt{0.000279} = 0.017$$

From (10), (11), (12):

$$\beta_M^B = \sqrt{1 - 0.87^2} = 0.49,$$

$$\beta_S^B = 0.87,$$

$$\tilde{\beta}^B_{ID,GNRL} = \beta^B_M \frac{w^B_{ID.GNRL}\sigma_{ID,GNRL}}{\hat{\sigma}_B} = 0.49 \frac{0.5 \times 0.03}{0.022} = 0.33,$$

$$\tilde{\beta}^B_{ID,AUTO} = \beta^B_M \frac{w^B_{ID.AUTO}\sigma_{ID.AUTO}}{\hat{\sigma}_B} = 0.49 \frac{0.2 \times 0.02}{0.022} = 0.09,$$

$$\tilde{\beta}^{B}_{ID.FOOD} = \beta^{B}_{M} \frac{w^{B}_{ID\ FOOD}\sigma_{ID\ FOOD}}{\hat{\sigma}_{B}} = 0.49 \frac{0.3 \times 0.04}{0.022} = 0.27,$$

$$\beta^{C}_{M} = \sqrt{1 - 0.88^{2}} = 0.47,$$

$$\beta^{C}_{S} = 0.88,$$

$$\tilde{\beta}^{C}_{US.GNRL} = \beta^{C}_{M} \frac{w^{C}_{US.GNRL}\sigma_{US.GNRL}}{\hat{\sigma}_{C}} = 0.47 \frac{0.2 \times 0.01}{0.017} = 0.06$$

$$\tilde{\beta}^{C}_{US.BFIN} = \beta^{C}_{M} \frac{w^{C}_{US\ BFIN}\sigma_{US\ BFIN}}{\hat{\sigma}_{C}} = 0.47 \frac{0.8 \times 0.02}{0.017} = 0.44$$

Finally, compute correlation between r^{B} and r^{C} using (13)

$$\begin{aligned} cor\left(r^{B}, r^{C}\right) &= 0.33 \times 0.06 \times 0.01 + 0.33 \times 0.44 \times 0.03 + 0.09 \times 0.06 \times 0.02 + 0.09 \times 0.44 \times 0.04 \\ &\quad + 2 \times 0.27 \times 0.06 \times 0.01 + 2 \times 0.27 \times 0.44 \times 0.02 \\ &= 0.017 \end{aligned}$$

2.3 Simulation of correlated equity returns

Obligors correlation matrix is used in the model to generate correlated samples of normally distributed random vectors of equity returns In this section we describe the procedure based on Cholesky decomposition of the correlation matrix of index returns

Suppose there are K obligors and their standardized equity returns are represented as (see (12) above):

$$R_{k} = \tilde{\beta}^{k}_{1} r_{1} + \ldots + \tilde{\beta}^{k}_{n} r_{n} + \beta^{k}_{S} r^{k}_{S}, \quad k = 1,\ldots,K. \qquad (14)$$

Denote by

$$\bar{R} = (R_{1},\ldots,R_{K})^{T}, \quad \bar{r} = \begin{pmatrix} \bar{r}_{M} \\ \bar{r}_{S} \end{pmatrix}_{(n+K)\times 1}, \quad \bar{r}_{M} = (r_{1},\ldots,r_{n})^{T}, \quad \bar{r}_{S} = \left(r^{1}_{S},\ldots r^{K}_{S}\right)^{T},$$

$$C_M = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{pmatrix}_{n\times n}, \quad \widetilde{C}_M = \begin{pmatrix} C_M & \mathbf{0} \\ \mathbf{0} & I_K \end{pmatrix}_{(n+K)\times(n+K)},$$

$$\mathrm{B} = \begin{pmatrix} \widetilde{\beta}_1^1 & \widetilde{\beta}_1^2 & \dots & \widetilde{\beta}_1^K \\ \widetilde{\beta}_2^1 & \widetilde{\beta}_2^2 & \dots & \widetilde{\beta}_2^K \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\beta}_n^1 & \widetilde{\beta}_n^2 & \dots & \widetilde{\beta}_n^K \end{pmatrix}_{n\times K}, \quad \Lambda = \begin{pmatrix} \beta_S^1 & 0 & \dots & 0 \\ 0 & \beta_S^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_S^K \end{pmatrix}_{K\times K}, \quad W = \begin{pmatrix} \mathrm{B} \\ \Lambda \end{pmatrix}_{(n+K)\times K},$$

where C_M is correlation matrix between indices and I_K is $K \times K$ identity matrix.

Then correlation matrix C of equity returns equals to

$$C = E\left(\overline{R}\overline{R}^T\right) = E\left(W^T \overline{r}\overline{r}^T W\right) = W^T E\left(\overline{r}\overline{r}^T\right) W = W^T \widetilde{C}_M W \tag{15}$$

Let us assume that matrix H_M is obtained by Cholesky decomposition of the matrix C_M :

$$C_M = H_M H_M^T. \tag{16}$$

Denote by

$$\widetilde{H}_M = \begin{pmatrix} H_M & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}_{(n+K)\times(n+K)},$$

$$H = W^T \widetilde{H}_M \tag{17}$$

(H is $K \times (n+K)$ matrix). Then

$$\widetilde{C}_M = \widetilde{H}_M \widetilde{H}_M^T$$

$$C = W^T \widetilde{H}_M \widetilde{H}_M^T W = HH^T \tag{18}$$

are Cholesky decompositions of C_M and C respectively.

For simulation of a sample of K dimensional normal vectors $\overline{R}$ with zero vector of means and covariance matrix C (standard deviations of coordinates of $\overline{R}$ equal 1) we first simulate $n+K$ dimensional vector of standard independent normal variables

$$\overline{x} = \begin{pmatrix} \overline{x}_M \\ \overline{x}_S \end{pmatrix} \sim N(0, I_{n+K}) ,$$

$$\overline{x}_M = (x_1, \ldots, x_n)^T \ , \ \overline{x}_S = (x_S^1, \ldots x_S^K)^T ,$$

and then compute $\overline{R}$ as

$$\begin{aligned} \overline{R} &= H\overline{x} = W^T \widetilde{H}_M \overline{x} \\ &= \begin{pmatrix} B^T & \Lambda \end{pmatrix} \begin{pmatrix} H_M & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \overline{x}_M \\ \overline{x}_S \end{pmatrix} = \begin{pmatrix} B^T & \Lambda \end{pmatrix} \begin{pmatrix} H_M \overline{x}_M \\ \overline{x}_S \end{pmatrix} = B^T H_M \overline{x}_M + \Lambda \overline{x}_S \end{aligned} \quad (19)$$

From (18) follows that $\overline{R}$ has distribution $N(0, C)$.

$$E(\overline{R}) = 0 ,$$

$$cov(\overline{R}) = E(H\overline{x}\overline{x}^T H^T) = HE(\overline{x}\overline{x}^T)H^T = HI_{n+K}H^T = HI_{n+K}H^T = C .$$

Equation (19) can be used for simulating correlated sample of equity returns in the model.

Appendix 3 to Part 1

Pricing Procedures

3.1 Pricing Procedure for Floating Rate Notes

Floating rate notes are designed so that the coupon payments depend on some current interest rate index (usually LIBOR or some other known rate), which is called reference rate This causes the value of a Floating Rate Note (FRN) to be close to par at all times. Credit risky FRNs pay coupons combined of reference rate plus scme spread over it, which depends on the credit rating of the issuer.

The notation is as follows

P – price of note,

r_i – expected future reference rate

y_t – spot long-term yields, used to discount the cash-flows at time t

s – stated credit spread, fixed under terms of the FRN

m – maturity of note in years

n – coupon payment frequency

Assuming flat structure of credit spreads, the price of the note per 100 of face value, is

$$P = \sum_{i=1}^{m \cdot n} \frac{100(r_i + s)/n}{(1 + y_i)^{t_i}} + \frac{100}{(1 + y_{m \cdot n})^{t_{m \cdot n}}}$$

Here, r_i is forward reference rate for the period between i-th and $(i+1)$-st coupon payments. Assuming the reference rate is annually compounded, the way to compute forward reference rate r_i is the following:

$$r_i = \left[\frac{(1 + R_{i+1})^{t_{i+1}}}{(1 + R_i)^{t_i}} \right]^{\frac{1}{t_{i+1} - t_i}} - 1$$

where R_i is the rate on the reference curve for time point t_i.

Within the current framework of CreditVaR I model, we will take existing current and forward corporate curves as a proxy for y_i, since they incorporate market required spread. The reference rate R and credit spread s should be supplied as parameters of the FRN.

Note that for computing present value of an FRN we use the supplied reference curve, but to compute forward 1-year value of an FRN, a forward reference curve has to be obtained. The rates r_i in the computation of forward value of an FRN are therefore "forward forward" rates.

3.2 Pricing Procedure for Interest Rate Swaps

Swaps can be characterized as the difference between two bonds, which can be either fixed or floating rate In the swap agreement, principal is not exchanged, which makes swap different from a bond from the point of view of credit risk. Another difference is that in the event of default by the counterparty, the sign of the value of a swap determines weather the other party pays or receives some recovery amount (this depends on netting rules)

Assumptions:

1) There is no more than one credit event per year
2) The credit event happens one year from evaluation date
3) Netting is done for positions with the same obligor and seniority
4) One recovery rate is applied to a netted position
5) Implementation allows different recovery models depending on the sign of exposure to a bank, including recovery rate 0 if the value of the swap to the bank is negative
6) A bank's credit events are not modelled (i e., the bank's credit rating is constant)

The notation is as follows

V – value of the swap to the bank,

B_1 – value of the bond modelling the cash flow received by the bank,

B_2 – value of the bond modelling the cash flow paid by the bank,

r_1 – recovery rate applied if bank receives in the event of default by counterparty,

r_2 – recovery rate applied if bank pays in the event of default by counterparty,

Q – notional pincipal in swap agreement.

The value of the swap to the bank then is:

$V = B_1 - B_2$

B_1 and B_2 are present (or forward 1 year) values of the underlying bonds, obtained by standard procedure for fixed or floating rate bonds. Ideally, swap bears credit risk only if the value of the swap to the financial institution is positive In practice, however, if counterparty defaults, it might not free the bank from liability to that counterparty (due to netting rules). Therefore, in case of default, the recovery amount is calculated as follows:

If $V \quad 0$, the recovered amount is $r_1 * V$,

If $V \quad 0$, the amount paid is $r_2 * V$.

Here, we don't take into consideration the event of default or credit rating migration by the bank. Recovery rates r_1 and r_2 are determined from legal considerations at the time of default. We can use Beta distribution again to simulate recovery rates r_1 and r_2 (in current implementation, we take $r_1 = r_2$). Alternatively, r_2 can be put to 0 if the swaps with negative exposure are excluded from the computation.

3.3 Pricing Procedure for Loan Commitments

A loan commitment is composed of a used portion and an unused portion. Used portion has a Loan (drawn funds) and a Letter of Credit (undrawn funds) parts. Interest (usually LIBOR plus spread) is paid on the Loan portion of a loan commitment, LIBOR plus LC fee is paid on the Letter of Credit part (throughout its lifetime), and an Unused Fee is paid on the unused

portion of the loan commitment. There is also a Facility Fee applied to the total loan commitment amount throughout its lifetime. A 1-time Upfront Fee paid on the total loan commitment amount at the initiation can be ignored in the CreditVaR I framework since it bears no credit risk.

The amount drawn at the risk horizon is closely related to the credit rating of the obligor. If the obligor deteriorates, it is more likely to withdraw additional funds. On the other hand, if its prospects improve, it is unlikely to need the extra borrowings.

Let T be the total (authorized) loan commitment amount, L – the amount drawn (in loans); LC – the amount in Letters of Credit, U – unused portion of the loan commitment.

We have that

$T = L + LC + U$

Let l be LIBOR and s be spread over LIBOR, so $(l + s)$ is the interest paid on the loan; lc – the fee paid on the letter of credit, uf – the fee paid on the unused portion; ff – facility fee applied to the total authorized amount of loan commitment. Then present value (in the CreditVaR framework) is computed as follows: we treat the loan and letter of credit parts as floating rate notes (because of the way interest is paid), facility fee and unused portion fee are valued as bonds with no principal repayment.

$$PV = \left[\sum_{i}^{mn} \frac{L \times (l_i + s)/n}{(1+y_i)^{t_i}} + \frac{L}{(1+y_{mn})^{t_{mn}}}\right] + \left[\sum_{i}^{mn} \frac{LC \times lc/n}{(1+y_i)^{t_i}}\right] + \left[\sum_{i}^{mn} \frac{U \times uf/n}{(1+y_i)^{t_i}}\right] + \left[\sum_{i}^{mn} \frac{T \times ff/n}{(1+y_i)^{t_i}}\right]$$

where y is the annualized rate taken from the corresponding current corporate curve used to discount cash flows (depends on credit rating of the obligor in the CreditVaR framework); n is the frequency of coupon payments, and m is the number of years to maturity. Forward

value (FV) is computed similarly, with y being the forward (1 year from now) rates taken from the corresponding corporate curve

If downgrade or default happen to a particular obligor, it is likely to withdraw additional funds from the loan commitment line. To model this we assume that there is an "expected" drawdown (ED), or "average commitment usage" if the obligor changes credit rating (see Credit Metrics Technical Document, p.45 for appropriate information):

Credit rating	Average commitment usage (ED)
Aaa	0.1%
Aa	1.6%
A	4.6%
Baa	20.0%
Ba	46.8%
B	63.7%
Caa	75.0%

Then in the case of any credit event (downgrade, upgrade, default) the used portion of a loan commitment is adjusted in the following way

Downgrade

Let $X\%$ ($Y\%$) is average commitment usage given in the Table above for the current (new) credit rating respectively, $X < Y$. Then new unused, loan and letter of credit portions are computed as

$$U_{new} = U\frac{1-Y}{1-X},$$

$$L_{new} = L + U - U_{new},$$

$$LC_{new} = LC,$$

- new used amount $T - U_{new}$ is proportionally distributed between loan amount L_{new} and letters of credit amount LC_{new}

Upgrade

In this case $X > Y$. New unused, loan and letter of credit portions are computed as

$$U_{new} = T - (L + LC)\frac{Y}{X},$$

$$L_{new} = \max(L - (U_{new} - U), 0)$$

$$LC_{new} = T - L_{new} - U_{new}.$$

Default

In the case of default it is assumed that used portion is the maximum of current used portion and 90% of the authorized amount T

$$L_{new} = \max(L + LC, 0.9 \times T),$$

$$LC_{new} = 0,$$

$$U_{new} = T - L_{new}.$$

Forward value of the loan commitment in the case of upgrade or downgrade is computed as

$$FV_{total} = FV + (L - L_{new})$$

In the case of default the loan commitment value is $L_{new} \times R$, where R is the recovery rate sampled from the Beta distribution (consistent with CreditVaR I framework), and total forward value is

$$FV_{total} = L_{new} \times R + (L - L_{new}).$$

Part 2 - CreditVaR Methodology: Credit Migration Process Under Two Probability Measures

As described in Part 1, the CreditVaR is a model for measuring and analyzing credit risk in a portfolio context The methodology used is based on credit migration analysis and on the assumption that an obligor's credit migration is driven by its asset value. Correlations between equity returns are used to compute joint probability distribution of obligors' credit migrations. Each obligor's standardized equity returns are decomposed into weighted average of market indices returns (multi-beta model) where the weights are specified to appropriately reflect the obligor's participation in the corresponding markets and to model obligor's idiosyncratic returns.

In the Monte Carlo simulation version the model calculates distribution of the portfolio values, statistical characteristics and risk measures defined as percentiles at given confidence levels (95%, 99%, etc). Simulation engine generates scenarios based on the "real world" probability distribution (P) of the risk factors. The distribution is assumed to be lognormal with parameters computed from the historical time series data. For each generated scenario the portfolio value is computed under the martingale probability measure (Q). The relationship between these two probability measures is derived from the assumptions about stochastic processes for index and stock returns. In particular, we derive the relationship between P and Q probabilities for obligors to migrate from one credit class to another including probabilities of default (transition matrices). This allows one to analyze portfolios of different instruments including credit derivatives.

In the CreditVaR framework it is assumed that credit migration process forms a discrete Markov chain with fixed moments of time $\widetilde{\mathbf{T}} = \{U_0 < U_1 < \ldots < U_i < \ldots\}$, $U_0 \geq 0$. In sections 1-5 of this Part we present mathematical models of the migration process and define probabilities under P and Q measures to migrate from one credit class at time U_i to another credit class at time U_{i+1}. Theoretical results of sections 1-5 are applied in sections 6-8 to compute P and Q transition probabilities from market credit spreads and credit migration historical data.

1. Stochastic processes for market index and stock returns under P measure

Suppose that $B = (B^1, B^2, \ldots, B^D)$ is a standard D-dimensional Brownian motion on a probability space $(\Omega, \mathbf{F}, P)$ ($B^1, B^2, \ldots, B^D$ are independent), where $\mathbf{F} = \{F_t, t \geq 0\}$ is a standard filtration of B.

Let $\mathbf{I} = \{I^1, I^2, \ldots, I^M\}$ be a set of all market indices. Denote by V_t^m the index I^m value at time $t \geq 0$. It is assumed that V_t^m satisfies stochastic differential equation (SDE) of the form

$$dV_t^m = V_t^m\left(\mu_t^m dt + \theta_{t,1}^m dB_t^1 + \theta_{t,2}^m dB_t^2 + \ldots + \theta_{t,D}^m dB_t^D\right), \quad t \geq 0$$

$$V_0^m = v^m, \qquad m = 1,2,\ldots,M, \tag{1.1}$$

where μ_t^m, $\theta_{t,i}^m$ are piecewise continuous deterministic functions of t, $i = 1,2,\ldots,D$.

Let us fix some time interval $[t_0, t_1]$ corresponding to two consecutive time steps from $\tilde{\mathbf{T}}$: $t_0 = U_l$, $t_1 = U_{l+1}$, $l \geq 0$. Denote $x^m = \ln(V_t^m) - \ln(V_{t_0}^m)$ be the index I^m log-return for the time interval $[t_0, t]$, $t_0 \leq t \leq t_1$. Then from Ito's lemma and (1.1) follows that

$$x_t^m = \int_{t_0}^{t}\left[\mu_s^m - \frac{1}{2}\left(\left(\theta_{s,1}^m\right)^2 + \left(\theta_{s,2}^m\right)^2 + \ldots + \left(\theta_{s,D}^m\right)^2\right)\right]ds$$
$$+ \int_{t_0}^{t} \theta_{s,1}^m dB_s^1 + \theta_{s,2}^m dB_s^2 + \ldots + \theta_{s,D}^m dB_s^D \tag{1.2}$$

for $t_0 \leq t \leq t_1$ with initial condition $x_{t_0}^m = 0$. In a matrix form (1.2) can be written as

$$x_t = a_t + \int_{t_0}^{t} \theta_s dB_s \ , \ t_0 \leq t \leq t_1 , \ x_{t_0} = 0 \ , \tag{1.3}$$

where

$x_t = \left(x_t^1, \ldots, x_t^M\right)^T$,

$$a = \left(a_t^1, \ldots, a_t^M\right)^T$$
$$= \left(\int_{t_0}^{t}\left[\mu_s^1 - \frac{1}{2}\left(\left(\theta_{s,1}^1\right)^2 + \left(\theta_{s,2}^1\right)^2 + \ldots + \left(\theta_{s,D}^1\right)^2\right)\right]ds, \ldots, \int_{t_0}^{t}\left[\mu_s^M - \frac{1}{2}\left(\left(\theta_{s,1}^M\right)^2 + \left(\theta_{s,2}^M\right)^2 + \ldots + \left(\theta_{s,D}^M\right)^2\right)\right]ds\right)^T.$$

$\theta_t = \left\|\theta_{t,i}^m\right\|_{M \times D}$,

$dB_s = \left(dB_s^1, \ldots, dB_s^D\right)^T$

Let us consider N obligors (issuers) $\mathbf{S} = \{S^1, S^2, \ldots, S^N\}$, and let H_t^n be an obligor S^n stock price at time t. We assume that H_t^n satisfies SDE:

$$dH_t^n = H_t^n\left(\alpha_t^n dt + \sigma_{t,1}^n dB_t^1 + \sigma_{t,2}^n dB_t^2 + \ldots + \sigma_{t,D}^n dB_t^D\right), \quad t \geq 0,$$

$$H_0^n = h^n, \qquad n = 1,2,\ldots,N, \tag{1.4}$$

where α_t^n, $\sigma_{t,i}^n$ are piecewise continuous deterministic functions of time, $i = 1,2,\ldots,D$. Then for the stock log-return $y_t^n = \ln(H_t^n) - \ln(H_{t_0}^n)$ one obtains

$$
\begin{aligned}
y_t^n = & \int_0^t \left[\alpha_s^n - \frac{1}{2}\left((\sigma_{s,1}^n)^2 + (\sigma_{s,2}^n)^2 + \ldots + (\sigma_{s,D}^n)^2 \right) \right] ds \\
& + \int_0^t \sigma_{s,1}^n dB_s^1 + \sigma_{s,2}^n dB_s^2 + \ldots + \sigma_{s,D}^n dB_s^D ,
\end{aligned} \tag{1.5}
$$

$$t_0 \le t \le t_1,\ y_{t_0}^n = 0 .$$

Similar to (1.3) this can be written as.

$$y_t = b_t + \int_{t_0}^t \sigma_s dB_s \ ,\ t_0 \le t \le t_1,\ y_{t_0} = 0 \ , \tag{1.6}$$

where

$$y_t = (y_t^1, \ldots, y_t^N)^T ,$$

$$
\begin{aligned}
b &= (b_t^1, \ldots, b_t^N)^T \\
&= \left(\int_0^t \left[\alpha_s^1 - \frac{1}{2}\left((\sigma_{s,1}^1)^2 + (\sigma_{s,2}^1)^2 + \ldots + (\sigma_{s,D}^1)^2 \right) \right] ds, \ldots, \int_0^t \left[\alpha_s^N - \frac{1}{2}\left((\sigma_{s,1}^N)^2 + (\sigma_{s,2}^N)^2 + \ldots + (\sigma_{s,D}^N)^2 \right) \right] ds \right)^T ,
\end{aligned}
$$

$$\sigma_t = \left\| \sigma_{t,i}^n \right\|_{N \cdot D}$$

2. Multi-beta model

In the CreditVaR framework correlations between stock returns are derived from the index return correlations using multi-beta model. This is linear regression model that represents stock returns as linear combinations of index returns and stock specific (idiosyncratic) components:

$$y_t^n = \beta_t^{n,0} + \beta_t^{n,1} x_t^1 + \beta_t^{n,2} x_t^2 + \ldots + \beta_t^{n,M} x_t^M + \varepsilon_t^n \ ,\ t_0 \le t \le t_1, \tag{2.1}$$

$n = 1,2,\ldots,N$,

where ε_t^n and x_t^m are uncorrelated: $cov(\varepsilon_t^n, x_t^m) = 0$, and $E(\varepsilon_t^n) = 0$, $m = 1,2,\ldots,M$, $n = 1,2,\ldots,N$. In the matrix form:

$$y_t = \beta_t^0 + \beta_t x_t + \varepsilon_t,$$

(2.2)

where $\beta_t^0 = (\beta_t^{1,0},\ldots,\beta_t^{N,0})^T$, $\beta_t = \|\beta_t^{n,m}\|_{N\cdot M}$ is a matrix of stock "betas", $\varepsilon_t = (\varepsilon_t^1,\ldots,\varepsilon_t^N)^T$ is a vector of idiosyncratic components which are uncorrelated with index returns:

$$\mathrm{cov}(\varepsilon_t, x_t) = \|\mathrm{cov}(\varepsilon_t^n, x_t^m)\|_{N\cdot M} = 0$$

(2.3)

(here and below expected value $E()$ and covariance $\mathrm{cov}()$ are conditional to σ-algebra F_{t_0})

It is assumed that $rank(\theta_t) = M \le D$ for all $t \in [t_0, t_1]$ (Note, that if $rank(\theta_t) < M \le D$ for some $t = \hat{t}$ then it can be shown that there exists time interval $[\bar{t},\bar{\bar{t}}] \subseteq [t_0, t_1]$, $\hat{t} \in [\bar{t},\bar{\bar{t}}]$ and index subset $\bar{\mathbf{I}} \subseteq \mathbf{I}$, $\bar{\mathbf{I}} \ne \emptyset$, such that for all indices in $\bar{\mathbf{I}}$ their log-returns can be represented as linear functions of index $\mathbf{I} \setminus \bar{\mathbf{I}}$ log-returns with time dependent deterministic coefficients. . Then index set $\mathbf{I}$ can be reduced to subset $\mathbf{I} \setminus \bar{\mathbf{I}}$ on $[\bar{t},\bar{\bar{t}}]$) Then from (1.8), (1.9) it follows that the $(M \times M)$-matrix $\theta_t\theta_t^T$ is nonsingular on $[t_0, t_1]$ and

$$\beta_t = \left(\int_{t_0}^{t} \sigma_s\theta_s^T ds\right)\left(\int_{t_0}^{t} \theta_s\theta_s^T ds\right)^{-1}$$

(2.4)

Indeed, from (1.3, (1.6) follows that

$$cov(x_t, x_t) = \int_{t_0}^{t} \theta_s \theta_s^T ds ,$$

(2.5)

$$cov(y_t, x_t) = \int_{t_0}^{t} \sigma_s \theta_s^T ds ,$$

(2.6)

Then (2.2) and (2.5) give

$$cov(y_t, x_t) = cov(\beta_t^0 + \beta_t x_t + \varepsilon_t, x_t) = \beta_t \, cov(x_t, x_t) = \beta_t \int_{t_0}^{t} \theta_s \theta_s^T ds$$

Therefore,

$$\int_{t_0}^{t} \sigma_s \theta_s^T ds = \beta_t \int_{t_0}^{t} \theta_s \theta_s^t ds$$

and (2.4) holds.

Equations (2.2) and (2.4) also give the covariance matrix between equity specific components:

$$\begin{aligned} cov(\varepsilon_t, \varepsilon_t) &= cov(y_t - \beta_t^0 - \beta x_t, y_t - \beta_t^0 - \beta x_t) \\ &= \int_{t_0}^{t} \sigma_s \sigma_s^T ds - \left(\int_{t_0}^{t} \sigma_s \theta_s^T ds \right) \left(\int_{t_0}^{t} \theta_s \theta_s^T ds \right)^{-1} \left(\int_{t_0}^{t} \theta_s \sigma_s^T ds \right) \end{aligned} \quad (2.7)$$

In the special case of constant volatility matrices θ and σ one has

$$cov(\varepsilon_t, \varepsilon_t) = (t - t_0)\sigma\left(J_D - \theta^T (\theta\,\theta^T)^{-1} \theta\right)\sigma^T ,$$

where J_D is $(D \times D)$ identity matrix In this case equity specific components ε_t^n are uncorrelated if and only if $\sigma\left(J_D - \theta^T (\theta\,\theta^T)^{-1} \theta\right)\sigma = \lambda$, where λ is a diagonal matrix. This means that orthogonal projections of the rows of stock volatility matrix σ on the linear subspace, which is an orthogonal compliment in R^D of the linear space generated by rows of

θ, are orthogonal. Note, that in this case covariance matrix of stock returns equals $cov(y_t, y_t) = (t - t_0)(\beta\theta\theta^T\beta^T + \lambda)$.

3. Credit migration model under probability measure P

It is assumed that at any time $U \in \tilde{\mathbf{T}}$ each obligor $S^n \in \mathbf{S}$ belongs to one of credit classes $\mathbf{R} = \{R_1, \ldots, R_K\}$, where R_K corresponds to default. Suppose that credit migration process forms a Markov chain on $\mathbf{R}$ with discrete times $\tilde{\mathbf{T}}$ and absorbing state R_K. Denote by R_U^n a credit class of obligor S^n at time $U \in \tilde{\mathbf{T}}$. Similar to section 1, we fix a time interval $[t_0, t_1]$ corresponding to two consecutive time steps from $\tilde{\mathbf{T}}$: $t_0 = U_l$, $t_1 = U_{l+1}$, $l \geq 0$. Suppose that $R_{t_0}^n = R_i$ and $p_{i,j}^n(t_0 \to t_1)$ is a probability of S^n to migrate to credit class R_j at time t_1:

$$p_{i,j}^n(t_0 \to t_1) = P\left(R_{t_1}^n = R_j \middle| R_{t_0}^n = R_i\right). \tag{3.1}$$

Denote by

$$T_n^P(t_0 \to t_1) = \left\| p_{i,j}^n(t_0 \to t_1) \right\|_{K \times K} \tag{3.2}$$

a transition matrix of S^n for the time interval $[t_0, t_1]$. Then

$$\begin{cases} \sum_{j=1}^{K} p_{i,j}^n(t_0 \to t_1) = 1, & j = 1, \ldots, K \\ p_{K,j}^n(t_0 \to t_1) = 0, & j = 1, \ldots, K-1 \\ p_{K,K}^n(t_0 \to t_1) = 1 \end{cases}.$$

For each nondefault state R_i and probabilities $p_{i,j}^n(t_0 \to t_1)$, $j = 1, \ldots, K$ define thresholds $h_{i,j}^n(t_0, t_1)$, $j = 1, \ldots, K-1$, for the standard normal distribution as follows:

$$\begin{aligned}
p^n_{i,1}(t_0 \to t_1) &= 1 - N\left(h^n_{i,1}(t_0,t_1)\right) \\
p^n_{i,2}(t_0 \to t_1) &= N\left(h^n_{i,1}(t_0,t_1)\right) - N\left(h^n_{i,2}(t_0,t_1)\right) \\
&\dots \\
p^n_{i,K-1}(t_0 \to t_1) &= N\left(h^n_{i,K-2}(t_0,t_1)\right) - N\left(h^n_{i,K-1}(t_0,t_1)\right), \\
p^n_{i,K}(t_0 \to t_1) &= N\left(h^n_{i,K-1}(t_0,t_1)\right)
\end{aligned} \tag{3.3}$$

where $N()$ is standard normal cumulative density function.

Let $w_t = Z^{-1}_{y,t}(y_t - E(y_t))$ be a vector of standardized stock returns with correlation matrix given by

$$cor(w_t, w_t) = Z^{-1}_{y,t} cov(y_t, y_t) Z^{-1}_{y,t} = Z^{-1}_{y,t}\left(\int_{t_0}^{t} \sigma_s \sigma_s^T ds\right) Z^{-1}_{y,t}, \tag{3.4}$$

where $Z_{y,t}$ is a $(N \times N)$ diagonal matrix with stock returns standard deviations

$$Z_{y,t} = \begin{pmatrix} \sqrt{\mathrm{var}(y_t^1)} & 0 & \cdots & 0 \\ 0 & \sqrt{\mathrm{var}(y_t^2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\mathrm{var}(y_t^N)} \end{pmatrix} \tag{3.5}$$

($\mathrm{var}(.)$ is a variance under P measure conditional to σ-algebra F_{t_0})

Similar to the CreditMetrics framework, the CreditVaR credit migration model is based on the assumption that credit class of the obligor S^n at time t_1 is defined by the value of its normalized stock return w_t^n and thresholds $h^n_{i,j}(t_0,t_1)$:

Assumption: Given $R^n_{t_0} = R_i$, $R^n_{t_1} = R_j$ if and only if $w^n_{t_1} \in \left(h^n_{i,j}(t_0,t_1), h^n_{i,j-1}(t_0,t_1)\right]$, where $i, j = 1,2,\ldots,K$, $h^n_{i,0}(t_0,t_1) = \infty$, $h^n_{i,K}(t_0,t_1) = -\infty$.

From the Assumption it follows that joint migration process for obligors can be defined using a joint probability distribution of their normalized stock returns. Indeed, suppose that $R^n_{t_0} = R_{i_n}$, $S^n \in \mathbf{S}$, $n = 1,\ldots,N$. Then normalized obligors' stock returns have N-dimensional normal distribution with zero means, standard deviations equal to 1 and correlation matrix (4 4) Denote by $f_N(x_1, x_2, \ldots, x_N)$ their joint density function. Then the joint probability that at time t_1 obligors will be in credit classes $R_{j_1}, R_{j_2}, \ldots, R_{j_N}$ respectively is given by

$$P\left(R^1_{t_1} = R_{j_1}, \ldots, R^N_{t_1} = R_{j_N} \middle| R^1_{t_0} = R_{i_1}, \ldots, R^N_{t_0} = R_{i_N}\right)$$
$$= \int_{h^1_{i_1,j_1}(t_0,t_1)}^{h^1_{i_1,j_1-1}(t_0,t_1)} \int_{h^2_{i_2,j_2}(t_0,t_1)}^{h^2_{i_2,j_2-1}(t_0,t_1)} \cdots \int_{h^N_{i_N,j_N}(t_0,t_1)}^{h^N_{i_N,j_N-1}(t_0,t_1)} f_N(x_1, x_2, \ldots, x_N) dx_1 dx_2 \ldots dx_N .$$

(3.6)

4. Stochastic processes for index and stock returns under Q measure

In this section we derive stochastic differential equations for the index and stock returns under the martingale probability measure Q.

Let r_t be an instantaneous risk-free interest rate at time t. We assume that r_t is a deterministic positive function of t. Then index values I^m_t, $m = 1,2,\ldots,M$, and stock prices H^n_t, $n = 1,2,\ldots,N$, for non-dividend-paying stocks (this assumption can be relaxed) satisfy following equations.

$$dV_t^m = V_t^m\left(r_t\,dt + \theta_{t,1}^m d\tilde{B}_t^1 + \theta_{t,2}^m d\tilde{B}_t^2 + \ldots + \theta_{t,D}^m d\tilde{B}_t^D\right), \qquad t \ge 0,$$

(4.1)

$$V_0^m = v^m, \qquad m = 1,2,\ldots,M,$$

$$dH_t^n = H_t^n\left(r_t\,dt + \sigma_{t,1}^n d\tilde{B}_t^1 + \sigma_{t,2}^n d\tilde{B}_t^2 + \ldots + \sigma_{t,D}^n d\tilde{B}_t^D\right), \qquad t \ge 0,$$

(4.2)

$$H_0^n = h^n, \qquad n = 1,2,\ldots,N,$$

where

$$\tilde{B}_t^d = B_t^d + \int_0^t \tau_s^d\,ds, \quad d = 1,\ldots,D, \tag{4.3}$$

are Brownian motions under the probability measure Q defined by the Radon-Nikodym derivative

$$\frac{dQ}{dP} = \exp\left(-\sum_{d=1}^{D}\int_0^t \tau_s^d\,dB_s^d - \frac{1}{2}\int_0^t |\tau_s|^2\,ds\right),$$

(4.4)

$$|\tau_t|^2 = \left(\tau_t^1\right)^2 + \left(\tau_t^2\right)^2 + \ldots + \left(\tau_t^D\right)^2,$$

where vector-function $\tau_t = \left(\tau_t^1, \tau_t^2, \ldots, \tau_t^D\right)^T$ solves equations:

$$\theta_{t,1}^m \tau_t^1 + \theta_{t,2}^m \tau_t^2 + \ldots + \theta_{t,D}^m \tau_t^D = \mu_t^m - r_t, \quad m = 1,2,\ldots,M,$$

(4.5)

$$\sigma_{t,1}^n \tau_t^1 + \sigma_{t,2}^n \tau_t^2 + \ldots + \sigma_{t,D}^n \tau_t^D = \alpha_t^n - r_t, \quad n = 1,2,\ldots,N$$

(4.6)

In the matrix form:

$$\begin{cases} \theta_t \tau_t = \tilde{\mu}_t \\ \sigma_t \tau_t = \tilde{\alpha}_t \end{cases},$$

(4.7)

where $\tilde{\mu}_t = (\mu_t^1 - r_t, \ldots, \mu_t^M - r_t)^T$, $\tilde{\alpha}_t = (\alpha_t^1 - r_t, \ldots, \alpha_t^N - r_t)^T$. We assume that solution τ_t of (4 7) exists for all $t \in [t_0, t_1]$

Similar to (1 5), (1 8) one computes index and stock log-returns from (4 1), (4 2) for the time interval $[t_0, t_1] = [U_l, U_{l+1}]$:

$$x_t = \hat{a}_t + \int_{t_0}^{t} \theta_s d\hat{B}_s \ , \ t_0 \le t \le t_1 \, , \ x_{t_0} = 0 \ ,$$

(4.8)

$$y_t = \hat{b}_t + \int_{t_0}^{t} \sigma_s d\hat{B}_s \ , \ t_0 \le t \le t_1 \, , \ y_{t_0} = 0 \ ,$$

(4.9)

where

$$\hat{a}_t = (\hat{a}_t^1, \ldots, \hat{a}_t^M)^T$$
$$= \left(\int_{t_0}^{t} \left[r_s - \frac{1}{2} \left((\theta_{s,1}^1)^2 + (\theta_{s,2}^1)^2 + \ldots + (\theta_{s,D}^1)^2 \right) \right] ds, \ldots, \int_{t_0}^{t} \left[r_s - \frac{1}{2} \left((\theta_{s,1}^M)^2 + (\theta_{s,2}^M)^2 + \ldots + (\theta_{s,D}^M)^2 \right) \right] ds \right)^T .$$

$$\hat{b}_t = (\hat{b}_t^1, \ldots, \hat{b}_t^N)^T$$
$$= \left(\int_{0}^{t} \left[r_s - \frac{1}{2} \left((\sigma_{s,1}^1)^2 + (\sigma_{s,2}^1)^2 + \ldots + (\sigma_{s,D}^1)^2 \right) \right] ds, \ldots, \int_{0}^{t} \left[r_s - \frac{1}{2} \left((\sigma_{s,1}^N)^2 + (\sigma_{s,2}^N)^2 + \ldots + (\sigma_{s,D}^N)^2 \right) \right] ds \right)^T .$$

$$d\hat{B}_s = (d\hat{B}_s^1, \ldots, d\hat{B}_s^D)^T .$$

Let $E^Q(.)$ $\mathrm{cov}^Q(.,.)$ be an expected value and covariance matrix under the probability measure Q, conditional to σ-algebra F_{t_0}. Then from (4.8), (4.9) follows that

$$E^Q(x_t) = \hat{a}, \tag{4.10}$$

$$E^Q(y_t) = \hat{b}, \tag{4.11}$$

$$cov^Q(x_t, x_t) = cov(x_t, x_t) = \int_{t_0}^{t} \theta_s \theta_s^T ds, \tag{4.12}$$

$$cov^Q(y_t, x_t) = cov(y_t, x_t) = \int_{t_0}^{t} \sigma_s \theta_s^T ds, \tag{4.13}$$

$$cov^Q(y_t, y_t) = cov(y_t, y_t) = \int_{t_0}^{t} \sigma_s \sigma_s^T ds \tag{4.14}$$

In particular,

$$\text{var}^Q(y_t^n) = \text{var}(y_t^n) \tag{4.15}$$

for all $n = 1,2,\ldots,N$ Moreover,

$$cov^Q(\varepsilon_t, x_t) = cov(\varepsilon_t, x_t) = 0 \ , \tag{4.16}$$

$$cov^Q(\varepsilon_t, \varepsilon_t) = cov(\varepsilon_t, \varepsilon_t), \tag{4.17}$$

but

$$E^Q(\varepsilon_t) = \hat{b}_t - \beta_t \hat{a}_t - \beta_t^0 \neq 0 \tag{4.18}$$

5. Credit migration model under Q measure

In section 3 we defined the relationship between credit migration events and movements of the normalized stock returns. This allows to compute probabilities of credit migrations for different credit classes under the Q measure.

Let obligor $S^n \in \mathbf{S}$ be in the credit class $R^n_{t_0} = R_i$ at time t_0. According to the assumption in section 3 its credit rating at time t_1 will be $R^n_{t_1} = R_j$ if and only if $w^n_{t_1} \in \left(h^n_{i,j}(t_0,t_1), h^n_{i,j-1}(t_0,t_1)\right]$, $j = 1,2,\dots,K$, where thresholds $h^n_{i,j}(t_0,t_1)$ are derived from the standard normal distribution. Then probability under Q measure for obligors $S^1, S^2, \dots, S^N$ to migrate from the credit classes $R^1_{t_0} = R_{i_1}, \dots, R^N_{t_0} = R_{i_N}$ at time t_0 to the credit classes $R^1_{t_1} = R_{j_1}, \dots, R^N_{t_1} = R_{j_N}$ at time t_1 respectively is

$$\begin{aligned} &Q\left(R^1_{t_1} = R_{j_1}, \dots, R^N_{t_1} = R_{j_N} \middle| R^1_{t_0} = R_{i_1}, \dots, R^N_{t_0} = R_{i_N}\right) \\ &= Q\left(w^n_{t_1} \in \left(h^1_{i_1,j_1}(t_0,t_1), h^1_{i_1,j_1-1}(t_0,t_1)\right], \dots, w^n_{t_1} \in \left(h^N_{i_N,j_N}(t_0,t_1), h^N_{i_N,j_N-1}(t_0,t_1)\right]\right) \end{aligned}$$

(5.1)

As follows from the definition of $w_t = \left(w^1_t, \dots, w^N_t\right)^T$ and (4 15)

$$\begin{aligned} w^n_t &= \frac{1}{\sqrt{\mathrm{var}\left(y^n_t\right)}}\left(y^n_t - E\left(y^n_t\right)\right) \\ &= \frac{1}{\sqrt{\mathrm{var}^Q\left(y^n_t\right)}}\left(y^n_t - E^Q\left(y^n_t\right)\right) + \frac{1}{\sqrt{\mathrm{var}\left(y^n_t\right)}}\left(E^Q\left(y^n_t\right) - E\left(y^n_t\right)\right) \\ &= \hat{w}^n_t + C^n_0(t_0,t) \end{aligned}$$

(5.2)

where

$$\hat{w}^n_t = \frac{1}{\sqrt{\mathrm{var}^Q\left(y^n_t\right)}}\left(y^n_t - E^Q\left(y^n_t\right)\right)$$

(5.3)

and $C^n_0(t_0,t)$ is a constant given by

$$C_0^n(t_0,t)=\frac{1}{\sqrt{var(y_t^n)}}\left(E^Q(y_t^n)-E(y_t^n)\right)$$
$$=\frac{1}{\sqrt{var(y_t^n)}}\int_{t_0}^{t}(r_s-\alpha_s^n)ds$$
$$=\frac{\int_{t_0}^{t}(r_s-\alpha_s^n)ds}{\left(\int_{t_0}^{t}\left((\sigma_{s,1}^n)^2+(\sigma_{s,2}^n)^2+\ldots+(\sigma_{s,D}^n)^2\right)ds\right)^{\frac{1}{2}}}$$

Then for the vector $\hat{w}_t=(\hat{w}_t^1,\ldots,\hat{w}_t^N)^T$ using (3.4), (4.14), (4.15) one has

$$\begin{aligned}\operatorname{cov}^Q(\hat{w}_t,\hat{w}_t)&=\operatorname{cov}^Q(w_t,w_t)\\&=E^Q\left(Z_{y,t}^{-1}(y_t-E(y_t))(y_t-E(y_t))^T Z_{y,t}^{-1}\right)\\&=Z_{y,t}^{-1}\operatorname{cov}^Q(y_t,y_t)Z_{y,t}^{-1}\\&=Z_{y,t}^{-1}\operatorname{cov}(y_t,y_t)Z_{y,t}^{-1}\\&=\operatorname{cov}(w_t,w_t).\end{aligned}\tag{5.4}$$

From equations (4.9), (5.3), (5.4) it follows that the random vector $\hat{w}_t=(\hat{w}_t^1,\ldots,\hat{w}_t^N)^T$ has a normal distribution under the probability measure Q with zero vector of means, covariance matrix $\operatorname{cov}^Q(w_t,w_t)=\operatorname{cov}(w_t,w_t)$ and probability density function $f_N(x_1,x_2,\ldots,x_N)$. Then $\hat{w}_t^n$ is a standard normal variable for each $n=1,\ldots,N$ and

$$\begin{aligned}q_{i,j}^n(t_0\to t_1)&=Q\left(w_{t_1}^n\in\left(h_{i,j}^n(t_0,t_1),h_{i,j-1}^n(t_0,t_1)\right)\right)\\&=Q\left(\hat{w}_{t_1}^n+C_0^n(t_0,t_1)\in\left(h_{i,j}^n(t_0,t_1),h_{i,j-1}^n(t_0,t_1)\right)\right)\\&=Q\left(\hat{w}_{t_1}^n\in\left(h_{i,j}^n(t_0,t_1)-C_0^n(t_0,t_1),h_{i,j-1}^n(t_0,t_1)-C_0^n(t_0,t_1)\right)\right)\\&=N\left(h_{i,j-1}^n(t_0,t_1)-C_0^n(t_0,t_1)\right)-N\left(h_{i,j}^n(t_0,t_1)-C_0^n(t_0,t_1)\right).\end{aligned}\tag{5.5}$$

Therefore, similar to transition probabilities $p_{i,j}^n(t_0\to t_1)$, transition probabilities $q_{i,j}^n(t_0\to t_1)$ under martingale measure Q are defined by shifted thresholds

$$g_{i,j}^{n}(t_0,t_1)=h_{i,j}^{n}(t_0,t_1)-\frac{\int_{t_0}^{t_1}(r_s-\alpha^n)ds}{\left(\int_{t_0}^{t_1}\left((\sigma_{s,1}^{n})^2+(\sigma_{s,2}^{n})^2+\ldots+(\sigma_{s,D}^{n})^2\right)ds\right)^{\frac{1}{2}}}$$

(5.6)

Note that transition probabilities $q_{i,j}^{n}(t_0\to t_1)$ depend on n and can be different for different obligors Equations (5.5), (5.6) show the relationship that exists between two transition matrices $T_n^P(t_0\to t_1)=\left\|p_{i,j}^{n}(t_0\to t_1)\right\|$ and $T_n^Q(t_0\to t_1)=\left\|q_{i,j}^{n}(t_0\to t_1)\right\|$

Similar to (3 6) for the Q probability of joint migrations we have

$$\begin{aligned}
&Q\left(R_{t_1}^{1}=R_{j_1},\ldots,R_{t_1}^{N}=R_{j_N}\middle|R_{t_0}^{1}=R_{i_1},\ldots,R_{t_0}^{N}=R_{i_N}\right)\\
&=Q\left(w_{t_1}^{n}\in\left(h_{i_1,j_1}^{1}(t_0,t_1),h_{i_1,j_1-1}^{1}(t_0,t_1)\right],\ldots,w_{t_1}^{n}\in\left(h_{i_N,j_N}^{N}(t_0,t_1),h_{i_N,j_N-1}^{N}(t_0,t_1)\right]\right)\\
&=Q\left(\hat{w}_{t_1}^{n}\in\left(g_{i_1,j_1}^{1}(t_0,t_1),g_{i_1,j_1-1}^{1}(t_0,t_1)\right],\ldots,\hat{w}_{t_1}^{n}\in\left(g_{i_N,j_N}^{N}(t_0,t_1),g_{i_N,j_N-1}^{N}(t_0,t_1)\right]\right)\\
&=\int_{g_{i_1,j_1}^{1}(t_0,t_1)}^{g_{i_1,j_1-1}^{1}(t_0,t_1)}\cdots\int_{g_{i_N,j_N}^{N}(t_0,t_1)}^{g_{i_N,j_N-1}^{N}(t_0,t_1)}f_N(x_1,x_2,\ldots,x_N)dx_1dx_2\ldots dx_N.
\end{aligned}\tag{5.7}$$

6. Construction of transition matrix under Q-measure over risk horizon

In this section we suggest, as an application of the theory developed in earlier sections, a practical way to obtain a Q transition matrix for the period $[t_0,t_1]$, $t_1=t_0+H$, where H denotes a risk horizon, typically 1 year. We suppose that we are provided with a P transition matrix and a set of Q default probabilities, one for each initial credit rating, over the period. The matrix obtained this way will be used in subsequent sections.

The main point here is that of implementation issue. In practice, the differences between default thresholds of the given P transition matrix and those determined by the given Q default probabilities may not be independent of the credit ratings, contrary to the theory (see (5 6)) This is the case usually because the P transition matrix is obtained empirically from past historical data while the Q default probabilities are obtained from current price data. Below, we make a modification to the given P transition matrix and then apply the theory to obtain a Q transition matrix over the risk horizon Note that the last column of such Q transition matrix will coincide with the given Q default probabilities.

First, we introduce some general notation. For obligor $S^n \in \mathbf{S}$ and observation times $t > s$, $s, t \in \tilde{\mathbf{T}}$, let $q_i^n(s \to t)$ denote the Q default probability of S^n over the period $[s,t]$ given its initial rating of R_i at time s, i.e.,

$$q_i^n(s \to t) = Q(R_t^n = D \mid R_s^n = R_i), \quad i = 1,2,\ldots,K.$$

Recall that the default state $R_K = D$ is absorbing so that $q_K^n(s \to t) = 1$ Put $\kappa = K - 1$, and package them in a κ-dimensional vector

$$\mathbf{q}^n(s \to t) = \begin{pmatrix} q_1^n(s \to t) \\ q_2^n(s \to t) \\ \vdots \\ q_\kappa^n(s \to t) \end{pmatrix};$$

we have $\mathbf{q}^n(s \to t) \geq \mathbf{0}$, where $\mathbf{0}$ denotes the κ-dimensional zero vector.

Now suppose we are given a P transition matrix $\overline{T}_P^n = \| \overline{p}_{i,j}^n \|$ for each individual obligor S^n, respectively, and Q default probabilities $\mathbf{q}^n(t_0 \to t_1)$ over the risk interval $[t_0, t_1]$. Put $q_i^n = q_i^n(t_0 \to t_1)$ and $z_i^n = N^{-1}(q_i^n)$, $1 \leq i \leq \kappa$. Let

$$C_n^n = C_n^n(t_0,t_1) = \frac{\int_{t_0}^{t_1}\left(r_s - \alpha_s^n\right)ds}{\left(\int_{t_0}^{t_1}\left(\left(\sigma_{s,1}^n\right)^2 + \left(\sigma_{s,2}^n\right)^2 + \ldots + \left(\sigma_{s,D}^n\right)^2\right)ds\right)^{\frac{1}{2}}},$$

as in Section 5. Define transition matrix $T_n^P = \| p_{i,j}^n(t_0,t_1) \|$ by

$$p_{i,j}^n(t_0,t_1) = \begin{cases} N(z_i^n + C_n^n), & j = K \\ \upsilon_i \bar{p}_{i,j}^n, & j < K \end{cases} \qquad (i \neq K) \tag{6.1}$$

where $\upsilon_i = \dfrac{1 - N(z_i^n + C_n^n)}{1 - \bar{p}_{i,K}^n}$, $p_{K,j}^n(t_0,t_1) = 0$ if $j < K$ and $p_{K,K}^n(t_0,t_1) = 1$. We have adjusted the last column of the given matrix $\bar{T}_P^n$ so that the new P default thresholds satisfy $N^{-1}\left(p_{i,K}^n(t_0,t_1)\right) = z_i^n + C_n^n$ for $i < K$, i.e., they differ from the Q default thresholds z_i^n by constant C_n^n. The other entries corresponding to the probabilities of migration into non-default states are then adjusted while maintaining, for each row, the same relative weights with respect to the survival probability, i.e.,

$$\frac{p_{i,j}^n(t_0,t_1)}{1 - p_{i,K}^n(t_0,t_1)} = \frac{\bar{p}_{i,j}^n}{1 - \bar{p}_{i,K}^n} \quad \text{for all } i,j < K. \tag{6.2}$$

Now, we apply the theory of change of measure to T_n^P to obtain the desired Q transition matrix over the risk horizon.

With $T_n^P = \| p_{i,j}^n(t_0,t_1) \|$ of (6.1), let $h_{i,\cdot}^n(t_0,t_1)$ be its thresholds. In particular, we have

$$h_{i,K}^n(t_0,t_1) = N^{-1}(p_{i,K}^n(t_0,t_1)) = z_i^n + C_n^n \tag{6.3}$$

Let $T_n^Q = \| q_{i,j}^n(t_0,t_1) \|$ be the corresponding transition matrix under Q; by (5.5) and (5.6), its (i,j)-th entries are defined by

$$q_{i,j}^n(t_0,t_1) = N(g_{i,j-1}^n(t_0,t_1)) - N(g_{i,j}^n(t_0,t_1)), \quad i,j = 1,2,\ldots,K \tag{6.4}$$

where $g^n_{i,j}(t_0,t_1) = h^n_{i,j}(t_0,t_1) - C^n_n(t_0,t_1)$. Using (6.3), (6.4) and $N^{-1}(-\infty) = 0$, we have

$$q^n_{i,K}(t_0,t_1) = N(h^n_{i,K}(t_0,t_1) - C^n_n) - N(-\infty) = N(z^n_i) = q^n_i(t_0 \to t_1)$$

for each $i = 1,2,\ldots,K$. Thus, by construction, the last column of Γ^Q_n fits exactly the given default probabilities under Q over the period $[t_0,t_1]$.

$$\begin{pmatrix} q^n_{1,K}(t_0,t_1) \\ q^n_{2,K}(t_0,t_1) \\ \vdots \\ q^n_{K,K}(t_0,t_1) \\ 1 \end{pmatrix} = \begin{pmatrix} q^n_1(t_0 \to t_1) \\ q^n_2(t_0 \to t_1) \\ \vdots \\ q^n_K(t_0 \to t_1) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{q}^n(t_0 \to t_1) \\ 1 \end{pmatrix} \tag{6.5}$$

In summary, a suitable adjustment is made to the given P transition matrix such that its last column is compatible with the given Q default probabilities under the theory, i.e., their thresholds differ by a constant C independent of initial credit rating class. For each row, the remaining entries in the P transition matrix are also adjusted by retaining same relative weights with respect to the survival probability. The constant C is then subtracted from other non-default thresholds of the modified P matrix to obtain the corresponding thresholds, and therefore the entries, of the desired Q matrix.

Note, that in most practical applications transition matrices for individual obligors are not available. Instead only one common for all obligors P transition matrix is given. It can be, for instance, transition matrix published by one of rating agencies like Moody's or S&P, or the internal bank's transition matrix. For such case following modification of the procedure above can be suggested. We assume that Q default probabilities $\mathbf{q}^n(t_0 \to t_1)$ are also obligor independent so that $\mathbf{q}^n(t_0 \to t_1) = \mathbf{q}(t_0 \to t_1)$. Let q_i be the i-th component of $\mathbf{q}(t_0 \to t_1)$, and put

$$z_i = N^{-1}(q_i),$$

$$\overline{w}_i = N^{-1}(\overline{p}_{i,K}),$$

$$C_0 = \frac{1}{K}\sum_{i=1}^{K}(\overline{w}_i - z_i).$$

Here, for C_0, we may want to average the differences $\overline{w}_i - z_i$ over only the i-th terms for which q_i and $\overline{p}_{i,K}$ are not too small (so that the thresholds are away from $-\infty$); also, instead of taking the average, one can choose C_0 which gives the least mean square error. Similar to (6.1), define the new transition matrix $T^P = \| p_{i,j}(t_0,t_1) \|$ by

$$p_{i,j}(t_0,t_1) = \begin{cases} N(z_i + C_0), & j = K \\ \upsilon_i \overline{p}_{i,j}, & j < K \end{cases} \qquad (i \neq K)$$

where $\upsilon_i = \dfrac{1 - N(z_i + C_0)}{1 - \overline{p}_{i,K}}$; $p_{K,j}(t_0,t_1) = 0$ if $j < K$ and $p_{K,K}(t_0,t_1) = 1$. Therefore, the last column of the given matrix $\overline{T}_P$ is adjusted in such way that the new P default thresholds satisfy $N^{-1}(p_{i,K}(t_0,t_1)) = z_i + C_0$, i.e., they differ from the Q default thresholds z_i by constant C_0 independent of i. Similar to (6.2) we have

$$\frac{p_{i,j}(t_0,t_1)}{1 - p_{i,K}(t_0,t_1)} = \frac{\overline{p}_{i,j}}{1 - \overline{p}_{i,K}} \quad \text{for all } i, j < K.$$

7. Forward transition matrices under Q

In this section, we suggest an algorithm that builds a series of transition matrices matching the given term structure of default probabilities over various maturities. Throughout this section, we assume that probabilities and transition matrices are under Q-measure. Such series will be useful, for example, when pricing credit derivatives via risk neutral valuation.

Recall that for each obligor $S^n \in \mathbf{S}$ we assume that its credit migration process observed at times in $\tilde{\mathbf{T}}$ forms a Markov chain on states $\mathbf{R}$ under Q-measure. We view $\tilde{\mathbf{T}}$ as 'universal' in that all observation times considered from now on are elements of $\tilde{\mathbf{T}}$. In practice, we have a particular finite subset of observation times known from beginning.

Let $\mathbf{T} = \{u_0 < u_1 < \cdots < u_B\}$ $(\subseteq \tilde{\mathbf{T}})$ be a set of observation times. By a *series of transition matrices on* $\mathbf{T}$, we mean a sequence of transition matrices over the periods

$$[u_0, u_1],\ [u_1, u_2],\ \ldots,\ [u_{B-1}, u_B]$$

With such series, the transition matrix over any period $[u_a, u_b]$ with $a < b$ is given by the product of transition matrices over $[u_a, u_{a+1}]$, ..., $[u_{b-1}, u_b]$, respecting the order.

Let $\mathbf{T}_0 = \{t_l = t_0 + lH \mid l = 0,1,2,\ldots,L\}$ be a set of 'standard' observation times, where H is the risk horizon; we assume that $\mathbf{T}_0 \subseteq \tilde{\mathbf{T}}$. For obligor $S^n \in \mathbf{S}$, suppose we are given a term structure of default probabilities under Q-measure. By appropriately interpolating if necessary, this means that we are given the Q default probabilities satisfying

$$0 \le \mathbf{q}^n(t_0 \to t_1) \le \mathbf{q}^n(t_0 \to t_2) \le \cdots \le \mathbf{q}^n(t_0 \to t_L) < \mathbf{e}, \tag{7.1}$$

where $\mathbf{e}$ is the κ-dimensional vector whose components are all equal to 1. For b, $0 < b \le L$, and an invertible transition matrix M over the interval $[t_0, t_b]$ define vectors $\mathbf{q}^n_M(t_b \to t_l)$ as follows

$$\begin{pmatrix} \mathbf{q}^n_M(t_b \to t_l) \\ 1 \end{pmatrix} = M^{-1} \begin{pmatrix} \mathbf{q}^n(t_0 \to t_l) \\ 1 \end{pmatrix} \tag{7.2a}$$

for $l \in \{b, b+1, \ldots, L\}$. Potentially, these vectors may have negative components because M^{-1} can have negative entries. However, if these vectors satisfy

$$0 = \mathbf{q}^n_M(t_b \to t_b) \le \mathbf{q}^n_M(t_b \to t_{b+1}) \le \cdots \le \mathbf{q}^n_M(t_b \to t_L) < \mathbf{e}, \tag{7.2b}$$

then we refer to them as the *time t_b (forward) default probabilities obtained from the initial term structure of default probabilities* (7.1) *using transition matrix M*. Note that the equality in the first relation of (7.2b) is equivalent to the condition that the first κ components of the last column of M equals $\mathbf{q}^n(t_0 \to t_b)$, i.e., it matches the initially given default probabilities for the period $[t_0, t_b]$.

We wish to construct certain series of Q transition matrices

$$T^Q_n(t_0 \to t_1),\ T^Q_n(t_1 \to t_2),\ \ldots,\ T^Q_n(t_{L-1} \to t_L) \tag{7.3}$$

on $\mathbf{T}_n$ such that the following proposition is satisfied with $Z_l = T^Q_n(t_{l-1} \to t_l)$.

Proposition: Given initially a term structure of time t_0 default probabilities satisfying (7.1), there is a set of invertible matrices Z_1, Z_2, ..., Z_L such that for each b, $0 < b \le L$, the time t_b forward default probabilities obtained from (7.1) using $M = Z_1 Z_2 \cdots Z_b$ satisfy (7.2b), i.e., they are non-negative and increasing.

Proof of Proposition. We shall produce Z_l's recursively. At each stage, we aim to keep the entries corresponding to the probabilities of non-default migrations as 'close', in terms of maintaining their relative weights (This can be replaced by any other reasonable preference. For example, instead of seeking to retain same relative weights amongst the non-default probabilities, one can skew them in the direction of default probabilities: if default probability has increased then one can increase the relative weights towards lower credit classes, etc.), to those of the transition matrix T_0^Q which was produced from the given data for $[t_0, t_1]$ in Section 6

For the initial interval $[t_0, t_1]$, let $Y_1 = T_0^Q$, its last column, $\begin{pmatrix} \mathbf{q}''(t_0 \to t_1) \\ 1 \end{pmatrix}$ by (6 7), matches the given default probabilities for the interval. If Y_1 is invertible and condition (7.2b) already holds for $M = Y_1$, then we set $T_0^Q(t_0 \to t_1) = Y_1$. More generally, consider the following 1-parameter family of matrices

$$Z_1(\alpha) = \alpha X_1 + (1-\alpha) Y_1, \quad 0 \le \alpha \le 1, \tag{7.4a}$$

where

$$X_1 = \begin{pmatrix} \mathrm{diag}(r_1^1, \cdots, r_\kappa^1) & \mathbf{q}''(t_0 \to t_1) \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} r_1^1 & & & q_1 \\ & \ddots & & \vdots \\ & & r_\kappa^1 & q_\kappa \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{7 4b}$$

with $q_i = q_i''(t_0 \to t_1) < 1$ and $r_i^1 = 1 - q_i''(t_0 \to t_1)$ for $i = 1,2,\ldots,\kappa$. The matrices $Z_1(\alpha)$ all have the same last column and in particular is independent of α and equal to that of Y_1.

Choose the "smallest" α, call it α_1, in (7 4a) such that vectors $\mathbf{q}_M''(t_1 \to t_l)$ defined by (7.2a) with invertible $M = Z_1(\alpha)$ satisfy (7 2b) with $h = 1$ there (in practice, we fix a positive

integer $l > 1$ from beginning, and restrict values of α in the set $A(l) = \{0, 1/l, 2/l, \cdots, (l-1)/l, 1\}$. Then α_1 is chosen to be the smallest α in $A(l)$ for which condition (7.2b) holds.) That such α_1 exists is clear since for $\alpha = 1$, the condition is satisfied. Thus, we have the time t_1 forward default probabilities

$$0 \le \mathbf{q}''(t_1 \to t_2) \le \mathbf{q}''(t_1 \to t_3) \le \cdots \le \mathbf{q}''(t_1 \to t_L) < \mathbf{e} \tag{7.5}$$

with respect to $M = Z_1(\alpha_1)$; here, subscript M is omitted. We set $T_n^{\wp}(t_0 \to t_1) = Z_1(\alpha_1)$.

Now, for subsequent periods $[t_l, t_{l+1}]$, $l \ge 1$, we proceed recursively to construct invertible transition matrices. Using the time t_1 forward default probabilities (7.5) as given, we can obtain in a similar way an invertible transition matrix Z_2 on $[t_1, t_2]$ whose last column is compatible with the (forward) default probability $\mathbf{q}''(t_1 \to t_2)$ and whose other entries have, in each row, the same relative weights as those of $T_n^{\wp}$ (or, as before, have the weights distributed according to some preference). Note that the time t_2 forward default probabilities obtained from the time t_1 default probabilities (7.5) using Z_2 of $[t_1, t_2]$ will be the same as the ones obtained from the time t_0 default probabilities (7.1) using Z_1Z_2 on $[t_0, t_2]$. Repeating the procedure for intervals $[t_2, t_3]$, $[t_3, t_4]$ and so on, we construct a desired series of transition matrices, (7.3), on $\mathbf{T}_n$.

The key point about the construction is that the series of transition matrices built up this way fits the given term structure of default probabilities over each maturity t_l, $0 < l \le L$, in that the cumulative transition matrix $Z_1Z_2 \cdots Z_l$ on the interval $[t_0, t_l]$ has (the first κ components of) the last column matching the given $\mathbf{q}''(t_0 \to t_l)$.

Remark There are many choices for such series We have suggested simply one method that is practical and reasonable in incorporating much information about the P transition matrix given for the initial period as well as preserving non-negativity and increasing of the forward default probabilities defined by the constructed transition matrices Other alternate methods for producing such series can be made available. Actually, in practice, one may prefer to be given as exogenous input a series of forward Q transition matrices in place of the initial term structure of Q default probabilities

8. Joint process under Q on $\mathbf{T}$ and application to pricing of credit swaps

In the previous section, we have built series of Q transition matrices on standard observation times $\mathbf{T}_0$ for single processes R_t^n. In application, for example in pricing of credit derivatives, we need to 'translate' this series to cover a *different* set of observation times $\mathbf{T} = \{ u_0 < u_1 < \cdots < u_n \}$ ($\mathbf{T}$ includes 'nonstandard' time steps corresponding to payment dates, times to instrument's maturity, etc , which need not belong to $\mathbf{T}_0$) for a *joint* process For this, it is sufficient to describe it for single processes, for then the joint migration Markov process will be determined by (5 7) on each period $[u_{k-1}, u_k]$ as the co-migration of the obligors are governed by the correlation among their asset returns We make the assumption, in lack of any better information, that this correlation is constant in all periods, but this too can be given as input and made different for different periods. For the single process, it essentially requires that we provide a way to 'interpolate' a transition matrix for a given period to define a transition matrix for a subinterval of that period We shall simply take a certain Taylor series expansion of the 'fractional' power of the transition matrix, the power

being proportional to the subinterval's length, and when necessary make a minor adjustment in some cases

Let Z be a transition matrix for the period $[t_a, t_{a+1}]$ and suppose $[u, v] \subset [t_a, t_{a+1}]$ is a proper subinterval, put $\theta = (v - u)/(t_{a+1} - t_a)$ Now consider the matrix Z^θ defined by

$$Z^\theta = \sum_{k=0}^{\infty} ((\theta))_k (Z - I)^k , \tag{8.1}$$

where $((\theta))_k = \theta(\theta - 1)\cdots(\theta - k + 1)/k!$, as in the Taylor expansion around the identity matrix I When series (8 1) converges, we may truncate it after a certain number of terms to evaluate Z^θ with a small error Note that a typical transition matrix, especially over a period that is not too long, is 'near' I. In actual implementation of (8.1), it may give rise to possibly negative entries, though small in size In this case, one makes an adjustment in the resulting matrix, for example by replacing such entries by 0 and decreasing the adjusted amount from the diagonal probability. In most cases, this procedure works well and is robust.

We summarize various constructions made thus far Starting with a P transition matrix on the risk horizon and an initial term structure of Q default probabilities we construct series of transition matrices for single processes on standard observation times. These are then translated to produce series on a prescribed set of observation times $\mathbf{T}$, first for the single processes and then for the joint process Of course, for single processes, the Q series thus obtained can be used to imply the corresponding P series if differences in their thresholds are provided

Application: We indicate how the constructed series of Q transition matrices can be used in evaluating credit swaps A credit swap on a corporate bond or a reference facility provides

protection to the holder of the reference bond from the loss of value when the bond issuer defaults The buyer of the contract makes a series of premium payments, contingent upon the survival of either the seller or the bond issuer. Here, the scheduled dates for the premium payments are used as observation times **T** and the joint process will be that of the asset returns of two obligors, namely the counterparty C and the issuer B of the reference facility.

The price of a credit swap can be computed as the expected value, under Q-measure, of the difference between fixed rate and contingent payments. Also, when valuing the credit swap for risk measurement, it is important to consider whether the counterparty is a seller or a buyer, as the two situations are not exactly the opposite of each other. The payoff function for the contingent payments can be given generically in the form

$$X_{CP} = \sum_{E} f_E I_E ; \tag{8.2}$$

the sum is over an exhaustive set $\{E\}$ of all possible, mutually exclusive events, I_E is the indicator function of event E, and f_E is the cashflow, usually of what was promised in the contract or some fraction of it due to seller's own default, when event E occurs Similarly, one has the payoff function X_{FP} for the fixed payments side (buyer)

Denoting by τ_C and τ_B, respectively, the random default times of the counterparty C (seller) and the reference party B, then typically E is an event with a specified range for values of τ_C and τ_B. For example,

$$E = (\text{ event that } u_{i-1} < \tau_B \leq u_i \text{ and } \tau_C > u_i \text{ })$$

where $u_{i-1} < u_i$ are two specified times, usually a pair of consecutive premium payment times Depending on E, f_E is a function involving notional amount or forward values of the reference facility, a strike price, recovery rates of B and C, etc. and depends on whether

premium payments are made in advance or in arrears. Both E and f_E can be made explicit according to the credit swap's specification.

The price of the contingent claim is the expected discounted cashflows under the risk neutral pricing measure Q. It is equal to

$$\sum_E d_E f_E\, Q(E). \tag{8.3}$$

Given a term structure of interest rates, the risk free discounting, $d_E f_E$, of cashflows f_E can be approximated reasonably in a straightforward manner. Therefore, the remaining key ingredient in pricing credit swaps is to evaluate $Q(E)$, where E usually are some joint events pertaining to the default times τ_C and τ_B. In practice, all the probabilities $Q(E)$ appearing in pricing formula (8 3) can be expressed in terms of the entries of the constructed transition matrices. We illustrate this, for example, for the following event:

$$E = (\text{ event that } u_0 < \tau_B \le u_{k-1} \text{ and } u_0 < \tau_C \le u_k)$$

Let $I_0 = (R_{i_0}, R_{j_0})$ be the initial ratings of (B,C) at time u_0 and B_k the rating of B at time u_k (and similarly for C_k, etc). Then, we have

$$\begin{aligned} Q(E) &= Q(B_{k-1} = D, C_k = D \mid I_0) \\ &= \sum_{j=1}^{K} Q(C_k = D, B_{k-1} = D, C_{k-1} = R_j \mid I_0) \\ &= \sum_{j=1}^{K} Q(C_k = D \mid C_{k-1} = R_j) Q(B_{k-1} = D, C_{k-1} = R_j \mid I_0) \end{aligned}$$

where, for the last equality, we have used the Markov property for both the joint and single processes. Note that the probabilities appearing in the final stage are entries of the transition matrices, over periods $[u_{k-1}, u_k]$ and $[u_0, u_{k-1}]$. Hence $Q(E)$ are easily attained from the transition matrices. This completes the final piece of computation in pricing credit swaps.

Part 3 – Stochastic Rates

1. Introduction

CreditVaR model with stochastic interest and FX rates (CreditVaR II) is an extension of the CreditVaR model described in the Part I (referred further as CreditVaR I). The chart in Figure 6 describes the implementation of CreditVaR II model. CreditVaR II is a purely simulation engine and relies on generating correlated risk factors without pre-generation (like CreditVaR I), and full re-pricing of the portfolio for each scenario. Outputs are Value-at-Risk and various other statistics for portfolio as a whole as well as standalone and marginal statistics for each instrument in the portfolio.

2. Simulation Methodology

2.1. Simulation of Base Curves

Let 0 be the current time and h be the risk horizon. Let $y(t,T)$ be the continuously compounded zero coupon rate seen at time t for a bond maturing at $t+T$. Note that T is the term to maturity, not an absolute time point. Let $f(t,T_1,T_2)$ be the forward rate seen at time t for a contract starting at $t+T_1$ and maturing at $t+T_2$. We assume that the zero curve maturity buckets are the same for all currencies and all credit ratings, and we denote the maturity terms by $B=\{t_1,t_2,\dots,t_b\}$, where $t_0=0<t_1<t_2<\cdots<t_b$. For convenience, we denote the zero curve as seen at time h by $\mathbf{y}=(y_1,y_2,\dots,y_b)=(y(h,t_1),y(h,t_2),\cdots,y(h,t_b))$ and the one seen at time zero by $\mathbf{y}^0=(y_1^0,y_2^0,\dots,y_b^0)=(y(0,t_1),y(0,t_2),\cdots,y(0,t_b))$. The zero rate for a term between two bucket points is determined by linear interpolation. (If $t<t_1$ or $t>t_b$ then the zero rate is set to y_1 or y_b, respectively.)

Our first objective is to generate samples of $\mathbf{y}$, which are correlated within itself, spreads, FX rates and equity returns. (Note that there is a vector $\mathbf{y}$ for each currency. We dropped the subscript for currency to avoid heavy notation.) Consider the forward rates seen at h, denoted by $\mathbf{x}=(x_1,x_2,\dots,x_b)=(f(h,t_0,t_1),f(h,t_1,t_2),\cdots,f(h,t_{b-1},t_b))$. $\mathbf{x}$ can be given by

$$x_1=y_1 \text{ and } x_i=\frac{y_i t_i-y_{i-1}t_{i-1}}{t_i-t_{i-1}} \text{ for } i>1. \tag{1}$$

Conversely,

$$y_i=\frac{1}{t_i}\sum_{k=1}^{i}x_k(t_k-t_{k-1}) \tag{2}$$

Written in matrix forms,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_h \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ \frac{t_1}{t_2} & \frac{t_2 - t_1}{t_2} & & & \\ \frac{t_1}{t_3} & \frac{t_2 - t_1}{t_3} & \frac{t_3 - t_2}{t_3} & & \\ & & \ddots & \ddots & \\ \frac{t_1}{t_h} & \frac{t_2 - t_1}{t_h} & & & \frac{t_h - t_{h-1}}{t_h} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_h \end{bmatrix} \tag{3}$$

Denoting the above transition matrix by **T**, then $\mathbf{x}' = \mathbf{T}^{-1}\mathbf{y}'$ where the inverse matrix is given explicitly as

$$\mathrm{T}^{-1} = \begin{bmatrix} 1 & & & & \\ -\frac{t_1}{t_2 - t_1} & \frac{t_2}{t_2 - t_1} & & & \\ & \ddots & \ddots & & \\ & & & -\frac{t_{h-1}}{t_h - t_{h-1}} & \frac{t_h}{t_h - t_{h-1}} \end{bmatrix} \tag{4}$$

The following method is proposed to generate interest rate scenarios. Let $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_h^0)$ $= (f(0, t_0, t_1), f(0, t_1, t_2), \cdots, f(0, t_{h-1}, t_h))$ be the forward rates seen at time 0. We assume that the forward rates $\mathbf{x}$ are lognormally distributed and that the log relative changes $\xi = (\xi_1, \xi_2, \ldots, \xi_h) = (\ln\frac{x_1}{x_1^0}, \ln\frac{x_2}{x_2^0}, \ldots, \ln\frac{x_h}{x_h^0})$ follow a multivariate normal distribution $\mathrm{N}(\mu, \Omega)$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_h)$ are drifts and Ω is the covariance matrix. Therefore, the forward rates are guaranteed to be positive.

To simulate the state of zero rates at time h in the future (risk horizon), we first generate a correlated normal sample $\xi = (\xi_1, \xi_2, \ldots, \xi_h)$ from the distribution as defined above, obtain **x**

$=(x_1^0 \exp\xi_1, \ldots, x_b^0 \exp\xi_b)$, then apply equation (1) to get continuously compounded zero rates y

2.2. Simulation of Spreads

Given a credit rating R_i, $i = 1, \ldots, K-1$, let s_i be the spread curve for R_i over that of the base curve y. Thus, $y + s_i$ is the zero curve for R_i rating. The restrictions on variability of the spreads are the following: a) the corporate curves should increase with decreasing the credit rating; b) the corporate curves should not overlap for any time point t. We propose the following scheme for generating stochastic credit spreads. In order to reduce the number of risk factors in the simulation and satisfy the restrictions above, we generate the spread curves by a combination of Monte Carlo simulation and linear interpolation. The risk factors involved in simulation of spreads are:

- Credit spread $s_1(t_1)$ for the highest credit rating for (typically Aaa) and for maturity equal to the first bucket point on the zero curve (t_1)
- incremental spreads $\Delta s_i(t_1) = s_i(t_1) - s_{i-1}(t_1)$, $i = 2, \ldots, K-1$ for maturity t_1
- $\Delta s_{K-1}(t_b) = s_{K-1}(t_b) - s_{K-1}(t_1)$, i.e., the difference between the spreads for the lowest credit rating for the longest and shortest maturity.

The simulation relies on the lognormal assumption for the above risk factors. After they are generated as a part of a simulation scenario, we obtain spread curves by the following procedure:

1. Obtain $s_i(t_1) = s_{i-1}(t_1) + \Delta s_i(t_1)$. Lognormal assumption on $\Delta s_i(t_1)$ ensures that spreads are increasing when credit rating is decreasing, i.e. $s_i(t_1) > s_{i-1}(t_1)$ for $i = 2,..R$

2. Obtain the $s_{K-1}(t_h) = s_{K-1}(t_1) + \Delta s_{K-1}(t_h)$. We assume that $\Delta s_{K-1}(t_h)$ has lognormal distribution. This way we make sure that the spread curve doesn't decrease with maturity for any scenario

3. Obtain the rest of the spreads for maturity t_h by partitioning the interval $[0, s_{K-1}(t_h)]$ proportionally to $s_i(t_1)$ by taking $s_i(t_h) = s_i(t_1)\dfrac{s_{K-1}(t_h)}{s_{K-1}(t_1)}$. This way, we get spreads that increase when credit rating declines for maturity t_h.

4. Obtain the interior of the spread curves by linear interpolation between $s_i(t_1)$ and $s_i(t_h)$.

The corporate zero curves at risk horizon are obtained by adding the appropriate spread to the base curves

2.3. Simulation of FX rates

The FX rates at risk horizon are assumed to be lognormally distributed.

2.4. Simulation of credit migrations

Credit migrations and defaults are simulated similarly to the methodology of CreditVaR I (see Part I). The assumption is that credit migrations are generated by the change in asset

value of the obligor, which is linked to industry participation and idiosyncratic component. In the framework of CreditVaR II, the migrations are based on the transition matrix, idiosyncratic obligor returns and the correlation of industry indices with the rest of the risk factors and within themselves.

2.5. Statistics

Statistics contain means and standard deviations for some risk factors (forward rates, spreads, FX rates) and the covariance matrix of all the risk factors including industry indices.

3. Input Data

Obligor data:

Information about obligors is organized into a database containing details of their credit ratings, industries, and countries.

Portfolio data:

Information about financial positions is organized into portfolios of exposures. It covers different types of instruments such as fixed income instruments, loans, commitments, letters of credit, etc.

Market data and transition probabilities:

These include yield curves, spread curves, foreign exchange rates, transition probabilities from one credit rating to another for different credit rating systems (Moody 8 states, Moody 18 states, S&P 8 states)

Statistics:

Statistics contain means and standard deviations for some risk factors (forward rates, spreads, FX rates) and the covariance matrix of all the risk factors including industry indices.

4. Risk Statistics and Measurement

CreditVaR II produces risk measurements that are similar to the ones produces by CreditVaR I. On the portfolio level, the following quantities are calculated:

- Present value and Forward value of the portfolio at risk horizon
- Distribution of the portfolio value at risk horizon and its mean, standard deviation, skewness and kurtosis
- Value-at-Risk numbers at various percentile levels with 95% confidence bands
- Expected Excession (or Expected Loss Given Default) of the percentile level

On the instrument level, the following is calculated and reported:

- Present value and Forward value of the instrument at risk horizon
- Mean and Standard Deviation of the instrument value at risk horizon
- Marginal Standard Deviation and Delta-Standard Deviation (see below) of the instrument for a given portfolio
- Capital amount (see below)
- Optionally: Standalone Value-at-Risk numbers at various percentile levels
- Marginal Standard Deviation of instrument
- Delta-Standard Deviation of instrument

5. Credit Migration Model

Credit quality migrations are modelled according to Credit VaR I methodology (Part 1)

6. Simulation Engine

First, the engine prepares the necessary data for simulation. Threshold levels of standardized asset returns representing credit rating changes are determined for each obligor in the portfolio using the transition probabilities; this part depends only on the obligor's credit class and not on the obligor itself. Relevant data is read from the database and is stored in the appropriate data structures in memory.

The scenarios are generated based on the covariance matrix and other statistics of the risk factors that are relevant for a particular portfolio. Each scenario is composed of the base curve part, spread curve part, FX part and credit migration part.

For each of these scenarios, portfolio has to be revalued. In CreditVaR I model it was done quickly using pre-processed prices of instrument. CreditVaR II does full revaluation of the portfolio, without cash flow bucketing or any portfolio compression, based on the set of generated risk factors. Each time a default occurs in scenario, i.e., when the sampled asset return value of an obligor is below the default threshold level, a random recovery rate is generated according to a beta-distribution whose defining parameters are governed by seniority associated to the obligor's instruments. Finally, we obtain a distribution of portfolio values and from it the relevant risk statistics.

7. Implementation

The Global Analytics CreditVaR II program utilizes the methodology described above. The core computational engine is implemented in C++ Currently, interface for Windows is being developed, using the ADO database technology. Windows version currently utilizes a Microsoft Access database. ADO technology allows CreditVaR II to be used with the majority of available databases.

It is noted that those skilled in the art will appreciate that various modifications of detail may be made to the preferred embodiments described throughout this specification, which modifications would come within the spirit and scope of the invention as defined in the following claims.

CLAIMS

We claim

1. A method of creating an adjusted credit risk transition matrix of probabilities of credit migration comprising the steps of:

- obtaining a transition matrix under probability measure (P),
- adjusting the (P) transition matrix to be consistent with default probabilities under martingale measure (Q) by making a column of default probabilities in the (P) transition matrix consistent with default probabilities calculated under (Q), and
- scaling the other entries in the adjusted (P) transition matrix to compensate for the adjustment while retaining the relative weights among non-default classes from the obtained (P) transition matrix in the adjusted (P) transition matrix.

2. A credit risk model for the analysis of complex credit instrument, comprising:

an obtained transition matrix under probability measure (P), and

an adjusted (P) transition matrix based on the obtained (P) transition matrix and consistent with default probabilities under measure (Q) by making a column of default probabilities in the adjusted (P) transition matrix consistent with default probabilities calculated under (Q),

wherein the other entries in the adjusted (P) transition matrix are scaled to compensate for the adjustment while retaining the relative weights among non-default classes from the obtained (P) transition matrix in the adjusted (P) transition matrix.

3 The credit risk model of claim 2, further comprising a Monte Carlo simulation engine to generate valuation scenarios based on probability distribution computed using the adjusted (P) transition matrix.

4 The model of claim 2 or 3, wherein the model is implemented in computer software on a computer readable medium

5 The model of claim 4, wherein the software is for use on compatible hardware

6 The model of claim 2 or 3, wherein the adjusted (P) transition matrix is used to price credit swaps

7 The method of claim 1, wherein the step of adjusting the obtained (P) transition matrix is performed in accordance with the following:

$\mathrm{R} = \{R_1, \ldots, R_K\}$	- are all credit classes
R_K	- corresponds to default state
$T^P = p_{i,j}(t_u, t_1)$	- adjusted (P) transition matrix for the time interval $[t_u, t_1]$
$p_{i,j}(t_u, t_1)$	- transition probability from credit class R_i to credit class R_j

$$p^u_{i,j}(t_u, t_1) = \begin{cases} N(z^u_i + C_u), & j = K \\ \upsilon_i \bar{p}^u_{i,j}, & j < K \end{cases} \qquad (i \neq K)$$

$p_{K,j}(t_u, t_1) = 0$ if $j < K$ and $p_{K,K}(t_u, t_1) = 1$

$$C_u = C_u(G) = \frac{1}{|G|} \sum_{i \in G} (\bar{w}_i - z_i)$$

$$\bar{w}_i = N^{-1}(\bar{p}_{i,K})$$

$z_i = N^{-1}(q_i)$

$\upsilon_i = \dfrac{1 - N(z_i + C_u)}{1 - \bar{p}_{i,K}}$

$\bar{p}_{i,K}$ - obtained (P) probability of default for the credit class R_i (from the obtained (P) transition matrix)

q_i - (Q) probability of default for the credit class R_i

G - a set of credit classes (can be the whole set $\mathrm{R} = \{R_1, \ldots, R_K\}$ or some subset

$|G|$ - denotes the number of elements in G, if G is empty, put $C_u = 0$

$N(\ )$ - standard normal cumulative density function.

8 The method of claim 1, wherein the step of adjusting the (P) transition matrix is performed in accordance with the following

$T_n^P = \| p_{i,j}^n(t_0, t_1) \|$ - adjusted (P) transition matrix for the time interval $[t_0, t_1]$ for obligor n

$p_{i,j}^n(t_0, t_1)$ - obligor's n transition probability to migrate from credit class R_i to credit class R_j

$$p_{i,j}^n(t_0, t_1) = \begin{cases} N(z_i^n + C_u^n), & j = K \\ \upsilon_i \bar{p}_{i,j}^n, & j < K \end{cases} \qquad (i \neq K)$$

$p_{K,j}^n(t_0, t_1) = 0$ if $j < K$ and $p_{K,K}^n(t_0, t_1) = 1$

$$C_0^n = C_0^n(t_0, t_1) = \frac{\int_{t_0}^{t_1} (r_s - \alpha_s^n) ds}{\sqrt{var(y_{t_1}^n)}}$$

$$z_t^n = N^{-1}(q_t^n)$$

$$\upsilon_t = \frac{1 - N(z_t^n + C_0^n)}{1 - \bar{p}_{i,K}^n}$$

$\bar{p}_{i,K}^n$ - obtained (P) probability of default for the obligor n over interval $[t_0, t_1]$ if it is in the credit class R_i at time t_0

q_i^n - (Q) probability of default for the obligor n over interval $[t_0, t_1]$ if it is in the credit class R_i at time t_0

r_t - instantaneous risk-free interest rate at time t

$y_{t_1}^n$ - obligor n stock log-return over interval $[t_0, t_1]$

α_t^n - obligor n drift function of time t

$N()$ - standard normal cumulative density function

9. A method of pricing a credit swap, comprising the steps of:
   - Obtaining a (P) transition matrix over a risk horizon $[t_0, t_1]$ and the (Q) default probabilities over the same period $[t_0, t_1]$;
   - Creating an adjusted (P) transition matrix and (Q) transition matrix over $[t_0, t_1]$,
   - Building a first series of forward (Q) transition matrices for standard observation times $t = t_1, t_2, t_3, \ldots$ from the (Q) transition matrix and an initial term structure of default probabilities under Q over $[t_0, t_1]$,

- Translating the first series to produce a second series, having premium payment dates as the observation times by working with 'fractional' powers of transition matrices for subintervals,
- Deriving a third series of transition matrices from the second series for co-migration on credit classes for pairs of obligors;
- Multiplying cash flow for each payment date by a risk-free discount factor and by probability contained in the third series and summing the computed series for all payment dates; and
- Subtracting the sum from a payoff function for fixed payments

10 A method of pricing a credit swap, comprising the steps of:

- Obtaining a (P) transition matrix over a risk horizon $[t_0, t_1]$ and the (Q) default probabilities over the same period $[t_0, t_1]$;
- Creating an adjusted (P) transition matrix and (Q) transition matrix over $[t_0, t_1]$;
- Building a first series of forward (Q) transition matrices for standard observation times $t = t_1, t_2, t_3, \ldots$ from the (Q) transition matrix and an initial term structure of default probabilities under Q over $[t_0, t_1]$;
- Deriving a third series of transition matrices from the first series for co-migration on credit classes for pairs of obligors;
- Translating the third series to produce a second series, having premium payment dates as the observation times by working with 'fractional' powers of transition matrices for subintervals;

- Multiplying cash flow for each payment date by a risk-free discount factor and by probability contained in the second series and summing the computed series for all payment dates; and
- Subtracting the sum from a payoff function for fixed payments.

11. A method of pricing a credit swap, comprising the steps of:

- Obtaining a (P) transition matrix over a risk horizon $[t_0, t_1]$ and the (Q) default probabilities over the same period $[t_0, t_1]$;
- Creating an adjusted (P) transition matrix and (Q) transition matrix over $[t_0, t_1]$;
- Building a second series of forward (Q) transition matrices for premium payment dates from the (Q) transition matrix and an initial term structure of default probabilities under Q over $[t_0, t_1]$;
- Deriving a third series of transition matrices from the second series for co-migration on credit classes for pairs of obligors;
- Multiplying cash flow for each payment date by a risk-free discount factor and by probability contained in the third series and summing the computed series for all payment dates; and
- Subtracting the sum from a payoff function for fixed payments.

12. A system for pricing a credit swap, comprising:

- Means for obtaining a (P) transition matrix over a risk horizon $[t_0, t_1]$ and the (Q) default probabilities over the same period $[t_0, t_1]$;

- Means for creating an adjusted (P) transition matrix and (Q) transition matrix over $[t_0, t_1]$,
- Means for, from the adjusted (P) transition matrix and an initial term structure of default probabilities under Q over $[t_0, t_1]$, building a first series of forward transition matrices for standard observation times $t = t_1, t_2, t_3, \ldots$;
- Means for translating the first series to produce a second series, having premium payment dates as the observation times by working with 'fractional' powers of transition matrices for subintervals;
- Means for deriving a third series of transition matrices from the second series for co-migration on credit classes for pairs of obligors,
- Means for discounting cash flow for each payment date over the period by a risk-free discount factor and probability contained in the third series, and summing the computed series for all payment dates; and
- Means for subtracting the sum from a payoff function for fixed payments.

13. A system for pricing a credit swap, comprising the steps of:
    - Means for obtaining a (P) transition matrix over a risk horizon $[t_0, t_1]$ and the (Q) default probabilities over the same period $[t_0, t_1]$;
    - Means for creating an adjusted (P) transition matrix and (Q) transition matrix over $[t_0, t_1]$;
    - Means for building a first series of forward (Q) transition matrices for standard observation times $t = t_1, t_2, t_3, \ldots$ from the (Q) transition matrix and an initial term structure of default probabilities under Q over $[t_0, t_1]$;

- Means for deriving a third series of transition matrices from the first series for co-migration on credit classes for pairs of obligors,
- Means for translating the third series to produce a second series, having premium payment dates as the observation times by working with 'fractional' powers of transition matrices for subintervals,
- Means for multiplying cash flow for each payment date by a risk-free discount factor and by probability contained in the second series and summing the computed series for all payment dates, and
- Means for subtracting the sum from a payoff function for fixed payments.

14 A system for pricing a credit swap, comprising the steps of

- Means for obtaining a (P) transition matrix over a risk horizon $[t_0, t_1]$ and the (Q) default probabilities over the same period $[t_0, t_1]$,
- Means for creating an adjusted (P) transition matrix and (Q) transition matrix over $[t_0, t_1]$,
- Means for building a second series of forward (Q) transition matrices for premium payment dates from the (Q) transition matrix and an initial term structure of default probabilities under Q over $[t_0, t_1]$;
- Means for deriving a third series of transition matrices from the second series for co-migration on credit classes for pairs of obligors;
- Means for multiplying cash flow for each payment date by a risk-free discount factor and by probability contained in the third series and summing the computed series for all payment dates; and

- Means for subtracting the sum from a payoff function for fixed payments.

15. Software on a computer readable medium for pricing a credit swap, comprising instructions to

- Obtain a (P) transition matrix over a risk horizon $[t_0, t_1]$ and the (Q) default probabilities over the same period $[t_0, t_1]$,
- Create an adjusted (P) transition matrix and (Q) transition matrix over $[t_0, t_1]$,
- Build a first series of forward (Q) transition matrices for standard observation times $t = t_1, t_2, t_3, \ldots$ from the (Q) transition matrix and an initial term structure of default probabilities under Q over $[t_0, t_1]$,
- Translate the first series to produce a second series, having premium payment dates as the observation times by working with 'fractional' powers of transition matrices for subintervals,
- Derive a third series of transition matrices from the second series for co-migration on credit classes for pairs of obligors,
- Multiply cash flow for each payment date by a risk-free discount factor and by probability contained in the third series and sum the computed series for all payment dates, and
- Subtract the sum from a payoff function for fixed payments.

16. A method of modelling stochastic interest rates, credit spreads and foreign exchange rates comprising the steps of:

- Computing obligors' credit ratings at risk horizon based on country/industry market index logreturns and idiosyncratic components;

- Computing base zero interest rates for all currencies and given maturities $t_0, t_1, \ldots, t_b$ from base forward rates $f(t_0,t_1), f(t_1,t_2), \ldots, f(t_{b-1},t_b)$ corresponding to time intervals $[t_0,t_1]$, $[t_1,t_2]$, …, $[t_{b-1},t_b]$;
- Computing credit spreads $s_i(t_1) = s_{i-1}(t_1) + \Delta s_i(t_1)$ for maturity t_1 and credit classes $i = 2,\ldots,K-1$ (by currency),
- Computing credit spread $s_{K-1}(t_b) = s_{K-1}(t_1) + \Delta s_{K-1}(t_b)$ for the lowest credit rating and longest maturity t_b (by currency);
- Computing the rest of the spreads for maturity t_b by partitioning the interval $[0, s_{K-1}(t_b)]$ proportionally to $s_i(t_1)$ by taking $s_i(t_b) = s_i(t_1)\dfrac{s_{K-1}(t_b)}{s_{K-1}(t_1)}$ (by currency);
- Computing spreads for other maturities $t_2, t_3, \ldots, t_{b-1}$ by linear or any other type of interpolation between $s_i(t_1)$ and $s_i(t_b)$ (by currency);
- Computing the corporate zero curves at risk horizon by adding the appropriate spread to the base curves (by currency), and
- Computing foreign exchange rates.

17. A credit risk model for the analysis of portfolios of credit instruments, comprising

- Means for computing obligors' credit ratings at risk horizon based on country/industry market index logreturns and idiosyncratic components;
- Means for computing base zero interest rates for all currencies and given maturities $t_0, t_1, \ldots, t_b$ from base forward rates $f(t_0,t_1), f(t_1,t_2), \ldots, f(t_{b-1},t_b)$ corresponding to time intervals $[t_0,t_1]$, $[t_1,t_2]$, …, $[t_{b-1},t_b]$;

- Means for computing credit spreads $s_i(t_1) = s_{i-1}(t_1) + \Delta s_i(t_1)$ for maturity t_1 and credit classes $i = 2,...K-1$ (by currency);
- Means for computing credit spread $s_{K-1}(t_b) = s_{K-1}(t_1) + \Delta s_{K-1}(t_b)$ for the lowest credit rating and longest maturity t_b (by currency);
- Means for computing the rest of the spreads for maturity t_b by partitioning the interval $[0, s_{K-1}(t_b)]$ proportionally to $s_i(t_1)$ by taking $s_i(t_b) = s_i(t_1)\frac{s_{K-1}(t_b)}{s_{K-1}(t_1)}$ (by currency);
- Means for computing spreads for other maturities $t_2, t_3, \ldots, t_{b-1}$ by linear or any other type of interpolation between $s_i(t_1)$ and $s_i(t_b)$ (by currency),
- Means for computing the corporate zero curves at risk horizon by adding the appropriate spread to the base curves (by currency); and
- Means for computing foreign exchange rates.

18. The credit risk model of claim 17, wherein, all risk factors (interest and foreign exchange rates, market index returns, idiosyncratic components) have a joint probability distribution so that credit migration process and interest and foreign exchange rates are correlated.

19. The credit risk model of claim 18, further comprising a Monte Carlo simulation engine to generate valuation scenarios based on Monte Carlo simulations of risk factors.

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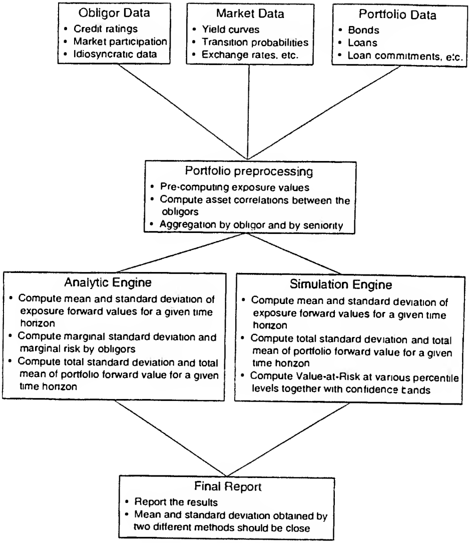
Obligor Data
• Credit ratings
• Market participation
• Idiosyncratic data
Market Data
• Yield curves
• Transition probabilities
• Exchange rates, etc.
Portfolio Data
• Bonds
• Loans
• Loan commitments, etc.
Portfolio preprocessing
• Pre-computing exposure values
• Compute asset correlations between the obligors
• Aggregation by obligor and by seniority
Analytic Engine
• Compute mean and standard deviation of exposure forward values for a given time horizon
• Compute marginal standard deviation and marginal risk by obligors
• Compute total standard deviation and total mean of portfolio forward value for a given time horizon
Simulation Engine
• Compute mean and standard deviation of exposure forward values for a given time horizon
• Compute total standard deviation and total mean of portfolio forward value for a given time horizon
• Compute Value-at-Risk at various percentile levels together with confidence bands
Final Report
• Report the results
• Mean and standard deviation obtained by two different methods should be close

Figure 1

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Portfolio Summary | Obligor Summary | Distribution | Marginal risk vs. Exposure

Portfolio name: Benchmark

Number of simulation runs: 25000

Number of exposures: 1863

Number of obligors: 207

Evaluation date: 6/18/98

Evaluation started: 6/18/98 10:08:45 AM

Evaluation finished: 6/18/98 10:09:34 AM

Analytic solution time: 0h 0m 0s

Simulation time: 0h 0m 40s

Present value: $919,215,199.16

Mean (analytic): $0.00

Mean (simulation): $868,465,536.31

Standard Deviation (analytic): $0.00

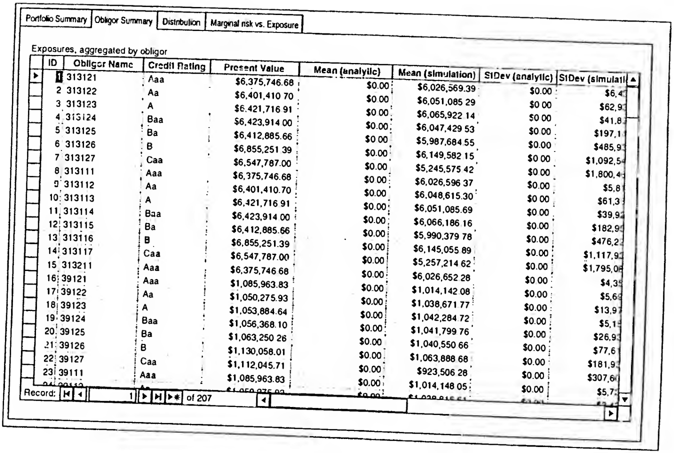
Standard Deviation (simulation): $25,567,995.38

Distribution Skewness (simulation) -1.45372019651998

Distribution Kurtosis (simulation) 2.93358666344538

Level	Percentile	95%C.I (upper bound)	95%C.I. (lower bound)	Expected Excession
10.00%	($34,121,155.12)	($34,942,044.31)	($33,478,332.21)	($57,010,862.68)
5.00%	($50,394,895.22)	($51,396,363 08)	($49,561,644 44)	($72,341,594.20)
1.00%	($85,747,382.61)	($88,080,488.21)	($84,022,958,66)	($104,057,914.97)
0.50%	($99,377,354.62)	($101,693,246.99)	($96,629,843.77)	($116,374,828.15)
0.10%	($128,296,322.87)	($135,226,674.88)	($123,675,928 96)	($141,899,201.05)

Figure 2



Portfolio Summary | Obligor Summary | Distribution | Marginal risk vs. Exposure

Exposures, aggregated by obligor

ID	Obligor Name	Credit Rating	Present Value	Mean (analytic)	Mean (simulation)	StDev (analytic)	StDev (simulati
1	313121	Aaa	$6,375,746.68	$0.00	$6,026,569.39	$0.00	$6,4
2	313122	Aa	$6,401,410 70	$0 00	$6,051,085 29	$0 00	$62,9
3	313123	A	$6.421,716 91	$0 00	$6,065,922 14	$0 00	$41.8
4	313124	Baa	$6,423,914 00	$0.00	$6,047,429 53	$0 00	$197,1
5	313125	Ba	$6,412,885.66	$0 00	$5,987,684 55	$0.00	$485,9
6	313126	B	$6,855,251 39	$0.00	$6,149,582 15	$0 00	$1,092,5
7	313127	Caa	$6,547,787.00	$0.00	$5,245,575 42	$0 00	$1,800,4
8	313111	Aaa	$6,375,746.68	$0 00	$6,026,596 37	$0.00	$5,8
9	313112	Aa	$6,401,410.70	$0.00	$6,048,615.30	$0 00	$61,3
10	313113	A	$6,421,716 91	$0.00	$6,051,085.69	$0 00	$39,9
11	313114	Baa	$6,423,914 00	$0.00	$6,066,186 16	$0.00	$182,9
12	313115	Ba	$6,412,885.66	$0.00	$5,990,379 78	$0.00	$476,2
13	313116	B	$6,855,251.39	$0.00	$6,145,055 89	$0.00	$1,117,9
14	313117	Caa	$6,547,787.00	$0.00	$5,257,214 62	$0.00	$1,795,0
15	313211	Aaa	$6,375,746 68	$0.00	$6,026,652 28	$0 00	$4,3
16	39121	Aaa	$1,085,963.83	$0.00	$1,014,142 08	$0.00	$5,6
17	39122	Aa	$1,050,275.93	$0.00	$1,038,671 77	$0.00	$13,9
18	39123	A	$1,053,884.64	$0.00	$1,042,284 72	$0.00	$5,1
19	39124	Baa	$1,056,368.10	$0.00	$1,041,799 76	$0.00	$26,9
20	39125	Ba	$1,063,250 26	$0.00	$1,040,550 66	$0.00	$77,6
21	39126	B	$1,130,058.01	$0.00	$1,063,888 68	$0.00	$181,9
22	39127	Caa	$1,112,045.71	$0.00	$923,506 28	$0.00	$307,6
23	39111	Aaa	$1,085,963.83	$0.00	$1,014,148 05	$0.00	$5,7

Record: 1 of 207

Figure 3

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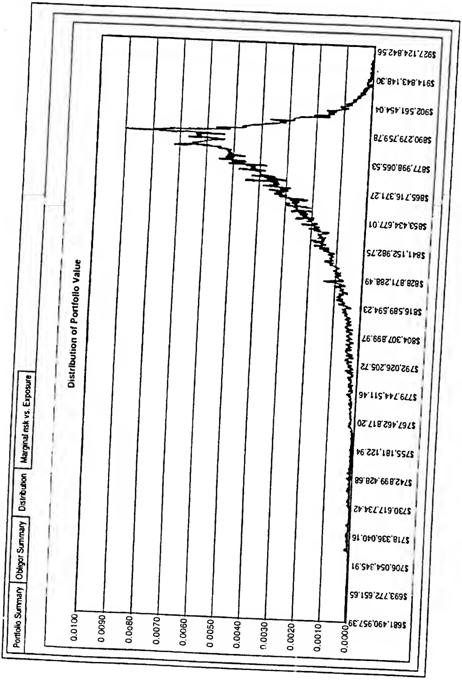
Portfolio Summary
Obligor Summary
Distribution
Marginal risk vs. Exposure
Distribution of Portfolio Value
0.0100
0.0090
0.0080
0.0070
0.0060
0.0050
0.0040
0.0030
0.0020
0.0010
0.0000
$681,490,957.39
$693,772,651.65
$706,054,345.91
$718,336,040.16
$730,617,734.42
$742,899,428.68
$755,181,122.94
$767,462,817.20
$779,744,511.46
$792,026,205.72
$804,307,899.97
$816,589,594.23
$828,871,288.49
$841,152,982.75
$853,434,677.01
$865,716,371.27
$877,998,065.53
$890,279,759.78
$902,561,454.04
$914,843,148.30
$927,124,842.56

Figure 4

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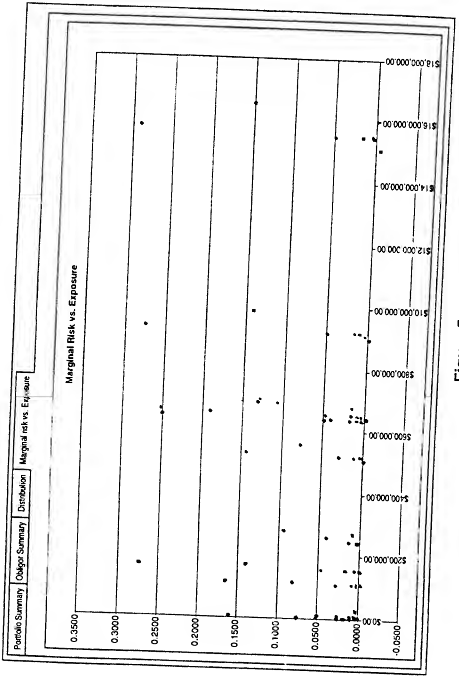
Portfolio Summary
Obligor Summary
Distribution
Marginal risk vs. Exposure
Marginal Risk vs. Exposure
0.3500
0.3000
0.2500
0.2000
0.1500
0.1000
0.0500
0.0000
-0.0500
$0.00
$200,000.00
$400,000.00
$600,000.00
$800,000.00
$10,000,000.00
$12,000,000.00
$14,000,000.00
$16,000,000.00
$18,000,000.00

Figure 5

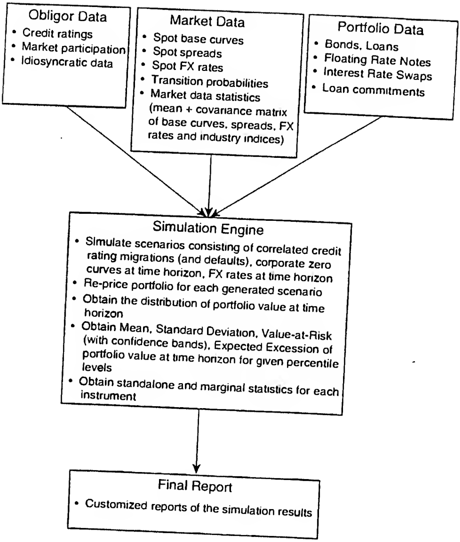
Obligor Data
• Credit ratings
• Market participation
• Idiosyncratic data
Market Data
• Spot base curves
• Spot spreads
• Spot FX rates
• Transition probabilities
• Market data statistics (mean + covariance matrix of base curves, spreads, FX rates and industry indices)
Portfolio Data
• Bonds, Loans
• Floating Rate Notes
• Interest Rate Swaps
• Loan commitments
Simulation Engine
• Simulate scenarios consisting of correlated credit rating migrations (and defaults), corporate zero curves at time horizon, FX rates at time horizon
• Re-price portfolio for each generated scenario
• Obtain the distribution of portfolio value at time horizon
• Obtain Mean, Standard Deviation, Value-at-Risk (with confidence bands), Expected Excession of portfolio value at time horizon for given percentile levels
• Obtain standalone and marginal statistics for each instrument
Final Report
• Customized reports of the simulation results

Figure 6